JENN
Training and Consultancy
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Province of the
EASTERN CAPE
EDUCATION

MATHEMATICS

GRADE 12

2024

LAST PUSH

TEACHER AND LEARNER CONTENT MANUAL

SCAN ME

INFORMATION SHEET

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$A=P(1+ni)$ $\quad$ $A=P(1-ni)$ $\quad$ $A=P(1-i)^n$ $\quad$ $A=P(1+i)^n$

$T_n=a+(n-1)d$ $\quad$ $S_n=\frac{n}{2}[2a+(n-1)d]$

$T_n=ar^{n-1}$ $\quad$ $S_n=\frac{a(r^n-1)}{r-1}$; $r\neq 1$ $\quad$ $S_\infty=\frac{a}{1-r}$; $-1<r<1$

$F=\frac{x[(1+i)^n-1]}{i}$ $\quad$ $P=\frac{[1-(1+i)^{-n}]}{i}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $\quad$ $M\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$

$y=mx+c$ $\quad$ $y-y_1=m(x-x_1)$ $\quad$ $m=\frac{y_2-y_1}{x_2-x_1}$ $\quad$ $m=\tan\theta$

$$(x-a)^2+(y-b)^2=r^2$$

In ΔABC: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ $\quad$ $a^2=b^2+c^2-2bc.\cos A$ $\quad$ $area\ \Delta ABC=\frac{1}{2}ab.\sin C$

$\sin(\alpha+\beta)=\sin\alpha.\cos\beta+\cos\alpha.\sin\beta$ $\quad$ $\sin(\alpha-\beta)=\sin\alpha.\cos\beta-\cos\alpha.\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha.\cos\beta-\sin\alpha.\sin\beta$ $\quad$ $\cos(\alpha-\beta)=\cos\alpha.\cos\beta+\sin\alpha.\sin\beta$

$\cos 2\alpha=\begin{cases}\cos^2\alpha-\sin^2\alpha\\1-2\sin^2\alpha\\2\cos^2\alpha-1\end{cases}$ $\quad$ $\sin 2\alpha=2\sin\alpha.\cos\alpha$

$\bar{x}=\frac{\sum fx}{n}$ $\quad$ $\sigma^2=\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$

$P(A)=\frac{n(A)}{n(S)}$ $\quad$ $P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$

$\hat{y}=a+bx$ $\quad$ $b=\frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$

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PAPER 1

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PAPER 1

ALGEBRA, EQUATIONS AND INEQUALITIES

TOPIC	ACTIVITY
Factorisation	Common factor, solve
	Transpose, factorise, solve
	Remove brackets, transpose, factorise, solve
Quadratic formula	Formula, substitution, answers correct to 2 decimal places/surd form
	Remove brackets, transpose/standard form, correct to 2 decimal places/surd form
Surds	Square both sides, solve, validate
	Transpose, square both sides, solve, validate
Simultaneous equations	Subject of the formula, substitution, standard form, factorise, solve, substitution for the other variable

Examination Guidelines

(a) Solving quadratic equations by completing the square will NOT be examined.

(b) Solving quadratic equations using the substitution method (k-method) is examinable.

(c) Equations involving surds that lead to a quadratic equation are examinable.

(d) Solution of non-quadratic inequalities should be seen in the context of functions.

(e) Nature of the roots will be tested intuitively with the solution of quadratic equations and in all the prescribed functions.

GOLDEN RULE!!!!!!: When transposing terms, the sign of that term changes.

Quadratic Equations

(a) Always make sure that the quadratic equation is in standard form first, i.e. on the other side of the equation, you are left with zero. That says it should be in the form $ax^2 + bx + c = 0$.

(b) You can now factorise or use quadratic formula, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to find values of x.

(c) When using quadratic formula, identify values of a (coefficient of x^2), b (coefficient of x) and c (a constant) first and write them down to avoid errors when substituting. If a t term having x is missing that means $b = 0$ and if a constant term is missing that means value of $c = 0$.

(d) Copy down the quadratic formula from the formula sheet and substitute the identified values in (c) above. ALWAYS open a bracket before substituting values of a, b and c and close the bracket after substitution.

(e) Substitute the information you have in your book exactly as it is in your calculator.

NB!

- ✓ **If the equation is in standard form and factorised, do not expand the brackets. Equate each bracket to zero and solve.**
- ✓ **Please be careful of rounding off if the final answer is in decimals.**
- ✓ **Some questions can be specific on which form the final answer must be. E.g. Surd form.**
- ✓ **When using a quadratic formula, there is no need to make coefficient of x^2 positive. Substitute the value of a together with its sign.**

Exponents	Same base, equate exponents, solve
	Laws of exponents, write as a power, factorise, equate exponents, solve
	Split, factorise, simplify
	Same bases, laws of exponents, equate exponents, solve
Inequalities	Critical values, sketch, answer
	Standard form, factorise, critical values, sketch, solve
	Remove brackets, standard form, factorise, critical values, sketch, solve
Nature of the roots	Use $b^2-4ac \geq 0$, solve
	Formula, substitution, $b^2-4ac \geq 0$, solve
	Rational, irrational, real/non-real

Quadratic Inequalities

(a) The quadratic inequality must be in standard form first, i.e. it should be in either form given: $ax^2 + bx + c > 0$, $ax^2 + bx + c \geq 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c < 0$. Note that, $ax^2 + bx + c > 0$ means we are looking for all possible values of x that will result with a positive answer, $ax^2 + bx + c \geq 0$, means we are looking for all possible values of x that will result with a positive answer or answer equal to zero, etc.

(b) Then, you can factorise or use quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ to find the critical values. The critical values are not solution/ may not be the only solution to the quadratic inequality, **BUT** the critical values are used to determine the solution to the given quadratic inequality.

(c) Use the critical values **AND** the standard form of a quadratic inequality to write down the solution. Solutions can be expressed in either interval form, number line form or inequality form.

Quadratic Inequality form	**Value of $\boldsymbol{a}$**	**Solution**
1. $ax^2 + bx + c > 0$	$a > 0$	$x < small\ C.V$ or $x > big\ C.V$ $x \in (-\infty\ ; small\ C.V)$ or $x \in (big\ C.V;\ \infty)$
2. $ax^2 + bx + c < 0$	$a > 0$	$small\ C.V < x < big\ C.V$ $x \in (small\ C.V;\ big\ C.V)$
3. $ax^2 + bx + c > 0$	$a < 0$	$small\ C.V < x < big\ C.V$ $x \in (small\ C.V;\ big\ C.V)$
4. $ax^2 + bx + c < 0$	$a < 0$	$x < small\ C.V$ or $x > big\ C.V$ $x \in (-\infty\ ; small\ C.V)$ or $x \in (big\ C.V;\ \infty)$

NB!

- ✓ **If the inequality is in standard form and factorised, you may expand the brackets to check sign of $\boldsymbol{a}$ OR simply observe the sign of $\boldsymbol{a}$ to get a correct solution. If $\boldsymbol{x}$ in both brackets have different signs, that says $\boldsymbol{a < 0}$ or $\boldsymbol{a \leq 0}$, otherwise $\boldsymbol{a > 0}$ or $\boldsymbol{a \geq 0}$. E.g. $\boldsymbol{(x-3)(4-x) > 0}$, $\boldsymbol{a < 0}$, then solution (3) above will be the one.**
- ✓ **When the inequality signs are $\boldsymbol{ax^2 + bx + c \leq 0}$ or $\boldsymbol{ax^2 + bx + c \geq 0}$, include equal sign in the inequality and replace curly brackets with square brackets, [ ]. When using number line method replace open circle with a shaded circle.**

Surd equations, leading to quadratic equations

(a) Separate the terms that are not affected by root sign from those affected by the root sign, in such a way that they are on different sides of the equation.

(b) Square the expressions on both sides to get rid of a root sign. Then the quadratic equation will come out.

(c) Solve it like it has been explained before.

(d) DO NOT forget to check if your final answers make LHS and RHS of the original question to be equal. Scratch out/ exclude the answer that yields different answers from both sides.

Exponential equations

Rules Of Exponents

1. $x^y x^z = x^{y+z}$	2. $\frac{x^y}{x^z} = x^{y-z}$	3. $(x^y)^z = (x^z)^y = x^{yz}$ $(xy)^z = x^z y^z$ $\frac{x^y}{z^y} = \left(\frac{x}{z}\right)^y$
4. $x^0 = 1$	5. $\frac{1}{x^y} = x^{-y}$ $\left(\frac{x}{z}\right)^{-y} = \left(\frac{z}{x}\right)^y$	6. $\sqrt[y]{x^z} = x^{\frac{z}{y}}$

An exponential equation is an equation where a variable (x or any other variable) is at the exponent. To solve for it,

(a) Simplify the equation until you have powers with the same base on either side of an equation ($a^{2x-1} = a^{-3}$). Then equate the exponents to solve for x.

OR

(b) Simplify the equation until you have powers with the same exponents on either side of an equation ($a^{2x} = b^{2x}$). Then divide both sides by either b^{2x} or a^{2x}. Then apply rule 3, $\left(\frac{a}{b}\right)^{2x} = 1$, and Applying rule 4, $2x = 0$.

(c) When solving exponential equations with rational exponents, E.g. If $x^3 = 8$, then $x = (8)^{\frac{1}{3}}$. Similarly, If $x^{\frac{3}{2}} = 27$, then $x = (27)^{\frac{2}{3}}$

(d) Equations of this nature $ae^{2x} + be^x + c = 0$ may be solved using K-method.

Nature of roots: Quadratic equations

The nature of roots depends on the value of the discriminant, Δ.

$\Delta = b^2 - 4ac$	**Roots**
$\Delta < 0$	**Non-real**
$\Delta \geq 0$	**Real**
$\Delta > 0$ *and perfect square*	**Real, Rational and Unequal**
$\Delta > 0$ *and not a perfect square*	**Real, Irrational and Unequal**
$\Delta = 0$	**Real, Rational and Equal**

PAPER A

QUESTION 1

1.1 Solve for x:

1.1.1 $3x^2 + 5x = 0$ (2)

1.1.2 $4x^2 + 3x - 5 = 0$ (answers correct to TWO decimal places) (3)

1.1.3 $(x-1)^2 - 9 \geq 0$ (4)

1.1.4 $5^{2x} - 5^x = 0$ (4)

1.1.5 $\frac{x}{\sqrt{20-x}} = 1$ (5)

1.2 Solve for x and y simultaneously:

$x + y = 9$ and $2x^2 - y^2 = 7$ (5)

1.3 Given: $P = (1-a)$ and $T = (1+a)(1+a^2)(1+a^4)..(1+a^{512})$

Determine the value of $P \times T$ in terms of a. (3)

[26]

PAPER B

QUESTION 1

1.1 Solve for x:

1.1.1 $x^2 + x - 12 = 0$ (3)

1.1.2 $3x^2 - 2x = 6$ (answers correct to TWO decimal places) (4)

1.1.3 $\sqrt{2x+1} = x - 1$ (4)

1.1.4 $x^2 - 3 > 2x$ (4)

1.2 Solve for x and y simultaneously:

$x + 2 = 2y$ and $\frac{1}{x} + \frac{1}{y} = 1$ (5)

1.3 Given: $2^{m+1} + 2^m = 3^{n+2} - 3^n$ where m and n are integers.

Determine the value of $m + n$. (4)

[24]

PAPER C

QUESTION 1

1.1 Solve for x:

1.1.1 $(3x-6)(x+2)=0$ (2)

1.1.2 $2x^2-6x+1=0$ (correct to TWO decimal places) (3)

1.1.3 $x^2-90>x$ (4)

1.1.4 $x-7\sqrt{x}=-12$ (4)

1.2 Solve for x and y simultaneously:

$$2x-y=2$$
$$xy=4$$ (5)

1.3 Show that $2.5^n-5^{n+1}+5^{n+2}$ is even for all positive integer values of n. (3)

1.4 Determine the values of x and y if: $\frac{3^{y+1}}{32}=\sqrt{96^x}$ (4)

[25]

PAPER D

QUESTION 1

(a) Solve for x:

(1) $x^2-5x=-6$ (3)

(2) $(3x+1)(x-4)<0$ (3)

(3) $\log_2(x+6)=1$ (2)

(4) $2x+\sqrt{x+1}=1$ (6)

(5) $12^{5+3x}=1$ (2)

(b) Solve for x and y:

$2x-y=8$ and $x^2-xy+y^2=19$ (7)

(c) The polynomial $x^{10}-2x^5+c$ is divisible by $x+1$. Calculate the value of c. (3)

(d) Determine the slope of the tangent to the graph of $y=x^2$ at the point (–1; 1). (2)

PAPER E

QUESTION 1

(a) Solve for x:

(1) $\frac{4x}{2}-\frac{2x+1}{3}=5$ (2)

(2) $(x-5)(x-6)\le 56$ (5)

(d) Given the equation $x^2+c=0$, where $-2<c<5$.

Give two values of c for which the roots of the equation are unequal and rational. (2)

(e) The roots of a quadratic equation are given by $x=\frac{-1\pm\sqrt{3-k}}{2}$.

Determine the value(s) of k for which the roots will be non-real. (2)

PAPER F

(a) Solve for x:

(1) $(x-1)^2=2(1-x)$ (4)

(2) $5^{-x}.5^{x-2}=\frac{25^{2x}}{5}$ (4)

Paper G

QUESTION 1

1.1 Solve for x:

1.1.1 $(2x-3)^2=1$ (3)

1.1.2 $2x^2+4x-7=0$ (Leave the answer in simplified surd form.) (4)

1.1.3 $x-\sqrt{2x-1}=2$ (6)

1.2 Consider the equation: $x^2+5xy+6y^2=0$

1.2.1 Calculate the values of the ratio $\frac{x}{y}$. (3)

1.2.2 Hence, calculate the values of x and y if $x+y=8$. (5)

1.3 The solutions of a quadratic equation are given by $x = \dfrac{-2 \pm \sqrt{2p+5}}{7}$

For which value(s) of p will this equation have:

1.3.1 Two equal solutions (2)

1.3.2 No real solutions (1)

PAPER H

QUESTION 1

1.1 Solve for x:

1.1.4 $(x+1)(4-x) > 0$ (3)

1.2 Given: $2^x + 2^{x+2} = -5y + 20$

1.2.1 Express 2^x in terms of y. (2)

1.2.2 How many solutions for x will the equation have if $y = -4$? (2)

1.2.3 Solve for x if y is the largest possible integer value for which $2^x + 2^{x+2} = -5y + 20$ will have solutions. (3)

PAPER I

QUESTION 1

1.1 Solve for x correct to 2 decimal places where necessary:

1.1.1 $(x-3)(x+4) = 18$ (4)

1.1.2 $x^2 = 6(x+2)$ (5)

1.1.3 $2 - \dfrac{1}{x} = \dfrac{3}{x+2}$ (6)

1.2 Solve for y if:

$$y^2 + 2y - \frac{8}{y^2 + 2y} = 7$$ (8)

1.3 Given: $3x^2 - 6px - 9p^2 = 0$

Solve for x in terms of p (6)

1.4 Solve for x if $2x^2 - 5x \geq 7$ (4)

QUESTION 2

2.1 Solve for x and y:

$$(2y+3)(x^2+4)=0$$ (3)

2.2 Solve simultaneously for a and b:

$2a-b=7$ and

$a^2+ab+b^2=7$ (8)

2.3 The area of a room is 20 m^2. If the length is increased by 3 m and the width is increased by 1 m, the room will double in area. Determine the original dimensions of the room. (5)

PATTERNS, SEQUENCES AND SERIES

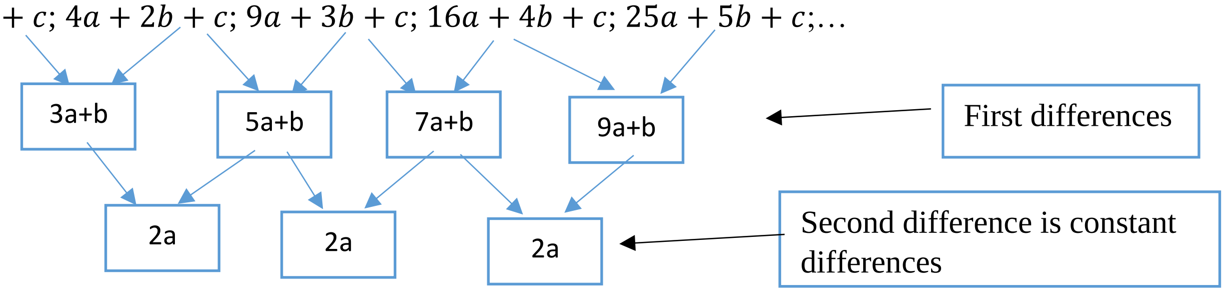
N.B $\boldsymbol{n} \in \boldsymbol{N}$

QUADRATIC NUMBER PATTERNS

The general formular: $T_n = an^2 + bn + c$

For any quadratic number pattern:

$a + b + c; 4a + 2b + c; 9a + 3b + c; 16a + 4b + c; 25a + 5b + c; \ldots$



N.B First differences of a quadratic number pattern form an arithmetic sequence

ARITHMETIC SEQUENCE AND SERIES

General term of an **arithmetic sequence**: $T_n = a + (n-1)d$

Where: a - first term of the sequence

d - common/constant difference

n – position of a term / number of terms

T_n – n^{th} term / last term

Sum formular for an **arithmetic series**: $S_n = \frac{n}{2}[2a + (n-1)d]$

OR

$$S_n = \frac{n}{2}[a + l] \qquad \text{where } l \text{ is the last term}$$

Proof:

$S_n = a + [a + d] + \ldots + [a + (n-2)d] + [a + (n-1)\, d] \ldots (1)$
rewrite equation (1) in reverse:
$S_n = [a + (n-1)d] + [a + (n-2)d] + \ldots + [a + d] + a \ldots (2)$

Adding equation (1) and equation (2)
$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + \ldots + [2a + (n-1)d] + [2a + (n-1)d]$
$2S_n = n \times [2a + (n-1)d]$
$S_n = \frac{n}{2}[2a + (n-1)d]$

GEOMETRIC SEQUENCE AND SERIES

General term of a **geometric sequence**: $T_n = ar^{n-1}$

Where: a - first term of the sequence

r - common/constant ratio

n – position of a term / number of terms

T_n – n^{th} term

Sum formular for a **geometric series**: $\boldsymbol{S_n = \frac{a(r^n-1)}{r-1}}$; $r \neq 1$

Proof:

$\boldsymbol{S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \ldots\ldots (1)}$

multiply equation (1) by r:

$\boldsymbol{rS_n = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \ldots\ldots\ldots (2)}$

Subtract equation (1) from equation (2)

$\boldsymbol{rS_n - S_n = -a + ar^n}$

$\boldsymbol{rS_n - S_n = ar^n - a}$

$\boldsymbol{S_n(r-1) = a(r^n - 1)}$

$$\boldsymbol{S_n = \frac{a(r^n-1)}{r-1}} \qquad ; r \neq 1$$

SUM TO INFINITY

Formular: $S_\infty = \frac{a}{1-r}$; $-1 < r < 1$

Sum to infinity exists for a convergent geometric series.

Convergence: $-1 < r < 1$

SIGMA NOTATION

$$\sum_{k=a}^{b} T_k$$

- T_k is the general term
- $number\ of\ terms = Top - Bottom + 1$

$$= b - a + 1$$

FOR ANY SUM FORMULAR

$$T_n = S_n - S_{n-1}$$

PAPER A

QUESTION 2

2.1 Consider the geometric series: $4+2+1+\frac{1}{2}+\ldots$

2.1.1 Does this series converge? Justify your answer. (2)

2.1.2 Calculate S_∞. (2)

2.2 Given: $\sum_{p=k}^{10} 3^{p-1} = 29\ 520$. Calculate the value of k. (5)

[9]

QUESTION 3

3.1 Consider the quadratic number pattern: $3\ ;\ 7\ ;\ 12\ ;\ \ldots$

3.1.1 Show that the general term of this number pattern is given by $T_n = \frac{1}{2}n^2 + \frac{5}{2}n$. (3)

3.1.2 What number must be added to T_{n-1} so that $T_n = 13\ 527$? (4)

3.2 Given an arithmetic sequence with $T_1 = 8$ and $T_2 = 11$.

3.2.1 Calculate the value of n if $T_n = 41$. (3)

3.2.2 A new arithmetic sequence P is formed using the term position and the term value of the given arithmetic sequence.
For the new sequence, $P_8 = 1$, $P_{11} = 2$ and so forth.

(a) Write down the value of P_{41}. (1)

(b) Calculate the value of the first term of the new arithmetic sequence. (4)

[15]

PAPER B

QUESTION 2

2.1 Given the arithmetic series: $7+12+17+\ldots$

2.1.1 Determine the value of T_{91} (3)

2.1.2 Calculate S_{91} (2)

2.1.3 Calculate the value of n for which $T_n = 517$ (3)

2.2 The following information is given about a quadratic number pattern:

$T_1 = 3$, $T_2 - T_1 = 9$ and $T_3 - T_2 = 21$

2.2.1 Show that $T_5 = 111$ (2)

2.2.2 Show that the general term of the quadratic pattern is $T_n = 6n^2 - 9n + 6$ (3)

2.2.3 Show that the pattern is increasing for all $n \in N$. (3)

[16]

QUESTION 3

3.1 Given the geometric series: $3 + 6 + 12 + ...$ to n terms.

3.1.1 Write down the general term of this series. (1)

3.1.2 Calculate the value of k such that: $\sum_{p=1}^{k} \frac{3}{2}(2)^p = 98\,301$ (4)

3.2 A geometric sequence and an arithmetic sequence have the same first term.

- The common ratio of the geometric sequence is $\frac{1}{3}$
- The common difference of the arithmetic sequence is 3
- The sum of 22 terms of the arithmetic sequence is 734 more than the sum to infinity of the geometric sequence.

Calculate the value of the first term. (5)

[10]

PAPER C

QUESTION 2

2.1 The first term of a geometric series is 14 and the 6^{th} term is 448.

2.1.1 Calculate the value of the constant ratio, r. (2)

2.1.2 Determine the number of consecutive terms that must be added to the first 6 terms of the series in order to obtain a sum of $114\,674$. (4)

2.1.3 If the first term of another series is 448 and the 6^{th} term is 14, calculate the sum to infinity of the new series. (3)

2.2 If $\sum_{p=0}^{k} \left(\frac{1}{3}p + \frac{1}{6}\right) = 20\frac{1}{6}$, determine the value of k. (5)

[14]

QUESTION 3

It is given that the general term of a quadratic number pattern is $T_n = n^2 + bn + 9$ and the first term of the first differences is 7.

3.1 Show that $b = 4$. (2)

3.2 Determine the value of the 60^{th} term of this number pattern. (2)

3.3 Determine the general term for the sequence of first differences of the quadratic number pattern. Write your answer in the form $T_p = mp + q$. (3)

3.4 Which TWO consecutive terms in the quadratic number pattern have a first difference of 157? (3)

[10]

PAPER D

QUESTION 4

A quadratic pattern has a second term equal to 1, a third term equal to -6 and a fifth term equal to -14.

4.1 Calculate the second difference of this quadratic pattern. (5)

4.2 Hence, or otherwise, calculate the first term of the pattern. (2)

QUESTION 7

The number pattern 1, 5, 11, 19, … is such that the second difference is constant.

7.1 Determine the 5^{th} number in the pattern. (1)

7.2 Derive a formula for the n^{th} number in the pattern. (7)

7.3 What is the 100^{th} number in the pattern? (3)

PAPER E

QUESTION 4

Consider the following quadratic number pattern: $-7 ; 0 ; 9 ; 20 ; \ldots$

4.1 Show that the general term of the quadratic number pattern is given by $T_n = n^2 + 4n - 12$. (4)

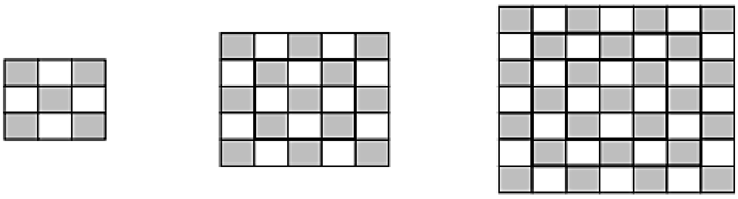
4.2 Which term of the quadratic pattern is equal to 128? (4)

4.3 Determine the general term of the first differences. (3)

4.4 Between which TWO terms of the quadratic pattern will the first difference be 599? (3)

QUESTION 5

Grey and white squares are arranged into patterns as indicated below.



Pattern 1 Pattern 2 Pattern 3

	Pattern 1	Pattern 2	Pattern 3
Number of grey squares	5	13	25

The number of grey squares in the n^{th} pattern is given by $T_n = 2n^2 + 2n + 1$.

5.1 How many white squares will be in the FOURTH pattern? (2)

5.2 Determine the number of white squares in the 157^{th} pattern. (3)

5.3 Calculate the largest value of n for which the pattern will have less than 613 grey squares. (4)

5.4 Show that the TOTAL number of squares in the n^{th} pattern is always an odd number. (3)

PAPER F

QUESTION 3

The first three terms of an arithmetic sequence are $2p - 3$; $p + 5$; $2p + 7$.

3.1 Determine the value(s) of p. (3)

3.2 Calculate the sum of the first 120 terms. (3)

3.3 The following pattern is true for the arithmetic sequence above:

$T_1 + T_4 = T_2 + T_3$

$T_5 + T_8 = T_6 + T_7$

$T_9 + T_{12} = T_{10} + T_{11}$

$\therefore T_k + T_{k+3} = T_x + T_y$

3.3.1 Write down the values of x and y in terms of k. (2)

3.3.2 Hence, calculate the value of $T_x + T_y$ in terms of k in simplest form. (4)

QUESTION 4

4.1 Given: $\sum_{k=1}^{\infty} 5(3^{2-k})$

4.1.1 Write down the value of the first TWO terms of the infinite geometric series. (2)

4.1.2 Calculate the sum to infinity of the series. (2)

4.2 Consider the following geometric sequence:

$\sin 30°; \cos 30°; \frac{3}{2}; ...; \frac{81\sqrt{3}}{2}$

Determine the number of terms in the sequence. (5)

PAPER G

QUESTION 3

3.1 The following geometric series is given:

$$2(3x-1)+2(3x-1)^2+2(3x-1)^3 \ \ldots$$

Determine the value(s) of x for which the series converges. (3)

3.2 The first two terms of a convergent geometric series are k and 6 respectively where $k \neq 0$.
The sum of the infinite series is 25.
Calculate the value(s) of k. (5)

3.3 Given the series:

$$(1\times 2)+(5\times 6)+(9\times 10)+(13\times 14)+\ldots+(81\times 82)$$

Write the series in sigma notation. (It is not necessary to calculate the value of the series). (4)

PAPER H

QUESTION 2

2.1 Consider the sequence: $\frac{1}{2}; 4; \frac{1}{4}; 7; \frac{1}{8}; 10; \ldots$

2.1.1 If the pattern continues in the same way, write down the next TWO terms in the sequence. (2)

2.1.2 Calculate the sum of the first 50 terms of the sequence. (7)

2.2 Consider the sequence: 8 ; 18 ; 30 ; 44 ; …

2.2.1 Write down the next TWO terms of the sequence, if the pattern continues in the same way. (2)

2.2.2 Calculate the n^{th} term of the sequence. (6)

2.2.3 Which term of the sequence is 330? (4)

2.3 Prove that for any arithmetic sequence of which the first term is a and the constant difference is d, the sum to n terms can be expressed as $S_n = \frac{n}{2}(2a + (n-1)d)$. (4)

QUESTION 3

Given the geometric series: $8x^2 + 4x^3 + 2x^4 + \ldots$

3.1 Determine the n^{th} term of the series. (1)

3.2 For what value(s) of x will the series converge? (3)

3.3 Calculate the sum of the series to infinity if $x = \frac{3}{2}$. (3)

PAPER I

QUESTION 2

A cyclist, preparing for an ultra cycling race, cycled 20 km on the first day of training. She increases her distance by 4 km every day.

2.1 On which day does she cycle 100 km? (3)

2.2 Determine the total distance she would have cycled from day 1 to day 14. (3)

2.3 Would she be able to keep up this daily rate of increase in distance covered indefinitely? Substantiate your answer. (2)

QUESTION 3

Consider the following sequence: 5; 12; 21; 32; ...

3.1 Write down the next term of the sequence. (1)

3.2 Determine a formula for the n^{th} term of this sequence. (5)

QUESTION 4

4.1 Prove that $a + ar + ar^2 + \ldots$ (to n terms) $= \dfrac{a(1-r^n)}{1-r}$ for $r \neq 1$ (6)

4.2 Given the geometric series $15 + 5 + \dfrac{5}{3} + \ldots$

4.2.1 Explain why the series converges. (2)

4.2.2 Evaluate $\sum_{n=1}^{\infty} 5(3^{2-n})$ (3)

4.3 The sum of the first n terms of a sequence is given by $S_n = 2^{n+2} - 4$

4.3.1 Determine the sum of the first 24 terms. (1)

4.3.2 Determine the 24^{th} term. (2)

4.3.3 Prove that the n^{th} term of the sequence is 2^{n+1}. (3)

FUNCTIONS AND GRAPHS

The domain is the set of allowed x-values which determine the range of y-values.

Important terminology

Domain:	the set of possible x-values	For all functions
Range:	the set of possible y-values	
Axis of symmetry:	an imaginary line that divides a graph into two mirror images of each other.	See the hyperbola and parabola
Maximum:	the highest possible y-value of a function.	See the parabola
Minimum:	the lowest possible y-value of a function.	
Asymptote:	an imaginary line that a graph approaches but never touches.	See the hyperbola and exponential function
Turning point:	The point at which a graph reaches its maximum or minimum value and changes direction.	See the parabola

The concepting of increasing and decreasing in functions: all functions

- The function is INCREASING when the value of y increases as x is increasing from left to right
  - **THE GRAPH GOES UP**

- The function is DECREASING when the value of y decreases as x is increasing from left to right
  - **THE GRAPH GOES DOWN**

LINEAR FUNCTIONS (STRAIGHT LINE)

The graph of $y = mx + c$

Standard form of linear function

WHERE

$m = gradient$

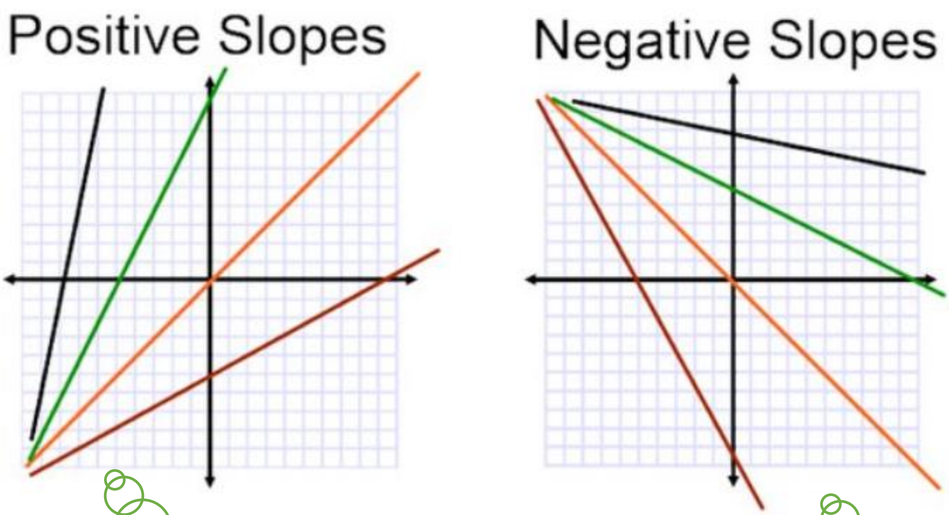
When $m > 0$ ***(gradient is positive)*** and

$m < 0$ ***(gradient is negative)***

Domain: $x \in R$

Range: $y \in R$

Shape



When $m > 0$

1. **The gradient is positive**
2. **The function is increasing**

When $m < 0$

3. **The gradient is negative**
4. **The function is decreasing**

HYPERBOLIC FUNCTIONS(HYPERBOLA)

The graph of $y = \frac{a}{x+p} + q$

Standard form of hyperbola

$$\boldsymbol{take\ note\ that\ y = \frac{2}{x-2} + 1}$$

$$\boldsymbol{= \frac{2}{x+(-2)} + 1}$$

The equations of asymptotes are $x = -p$ ***(vertical asymptote)*** and

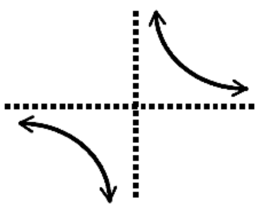
$y = q$ ***(horizontal asymptote)***

Domain: $x \in R, x \neq -p$

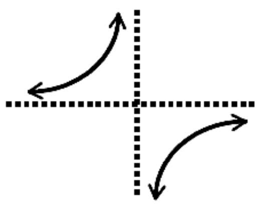
Range: $y \in R, y \neq q$

Shape

If $a > 0$ then the graph decreases for all $x < 0$ or $x > 0$.



If $a < 0$ then the graph increases for all $x < 0$ or $x > 0$.



The equations of the axis of symmetry

The hyperbola has two equations of symmetry

$\boldsymbol{m = 1}$	$\boldsymbol{m = -1}$
$\boldsymbol{y = x + c}$	$\boldsymbol{y = -x + c}$

N.B the equations of the axis of symmetry of the hyperbola passes through the point of intersection of asymptotes $(-p; q)$

In general, for the hyperbola, the equations of the axis of symmetry are given by the following formulae:

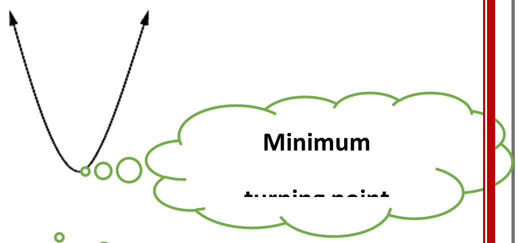
$m = 1$	$m = -1$
$y = (x + p) + q$ $\therefore y = x + p + q$	$y = -(x + p) + q$ $\therefore y = -x - p + q$

N.B Ensure that the hyperbola is in standard form before applying the formula

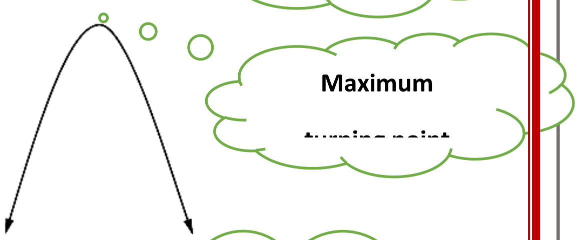
QUADRATIC FUNCTION(PARABOLA)

The graph of $y = a(x + p)^2 + q$

If a is positive, i.e. a > 0, then the shape of the graph is ☺.



If a is negative, i.e. a < 0, then the shape of the graph is ☹.



The graph has the axis of symmetry at $x = -p$

N.B

1. q is the minimum of the parabola when $a > 0$

2. q is the maximum of the parabola when $a < 0$

The graph has the turning point by $(-p\ ;\ q)$

Domain: $x \in R$

Range: $y \geq q$ 😊 **(WHEN a > 0)**

or

$y \leq q$ 😟 **(WHEN a < 0)**

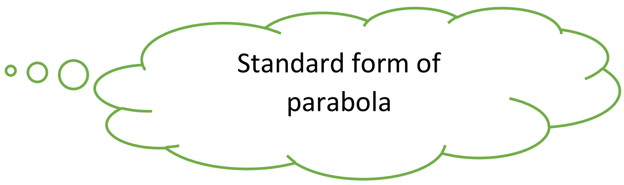
N.B The parabola changes from increasing to decreasing or decreasing to increasing at the turning point.

<u>**when a > 0**</u>
1. *The graph increases for*: $x > -p$
2. *The graph decrease for*: $x < -p$

<u>**when a < 0**</u>
1. *The graph increases for*: $x < -p$
2. *The graph decrease for*: $x > -p$

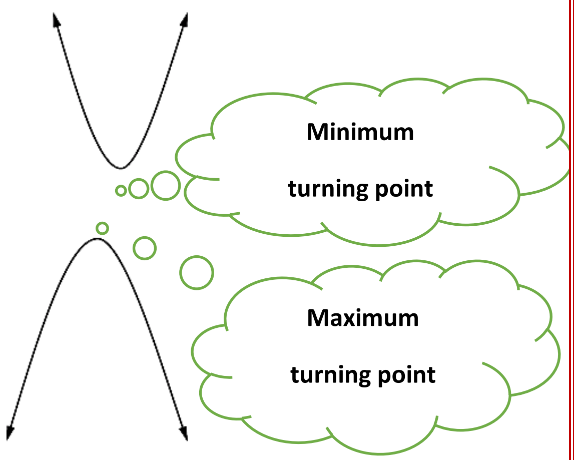
The quadratic function can also be represented in the form:

$y = f(x) = ax^2 + bx + c$



If a is positive, i.e. a > 0, then the shape of the graph is ☺.

If a is negative, i.e. a < 0, then the shape of the graph is ☹.



The graph has the axis of symmetry at $x = -\frac{b}{2a}$

The graph has the turning point by $(-\frac{b}{2a}\ ;\ f(-\frac{b}{2a}))$

Domain: $x \in R$

Range: $y \geq f(-\frac{b}{2a})$ 😊 **(WHEN a > 0)**

or

$y \leq f(-\frac{b}{2a})$ 😟 **(WHEN a < 0)**

N.B

1. $\boldsymbol{f\left(-\frac{b}{2a}\right)}$ **is the minimum of the parabola when** $\boldsymbol{a > 0}$

2. $\boldsymbol{f\left(-\frac{b}{2a}\right)}$ **is the maximum of the parabola when** $\boldsymbol{a < 0}$

<u>EXPONENTIAL FUNCTION</u>

Standard form of exponential function

The graph of $\boldsymbol{y = a.b^{x+p} + q}$ **where** $\boldsymbol{b}$ **> 0 and** $\boldsymbol{b \neq 1}$

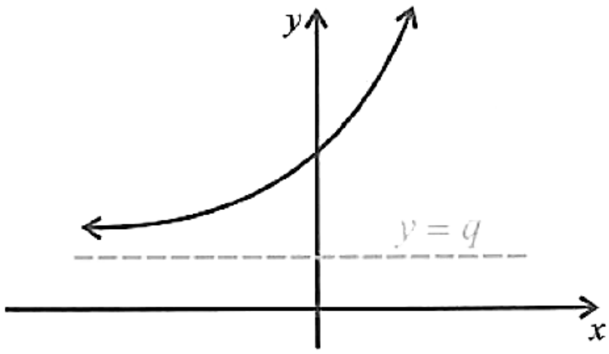
The equation of an asymptote is $y = q$ ***(horizontal asymptote)***

Domain: $x \in R$

Range: $y > q$ *[if a > 0]* or $y < q$ *[if a < 0]*

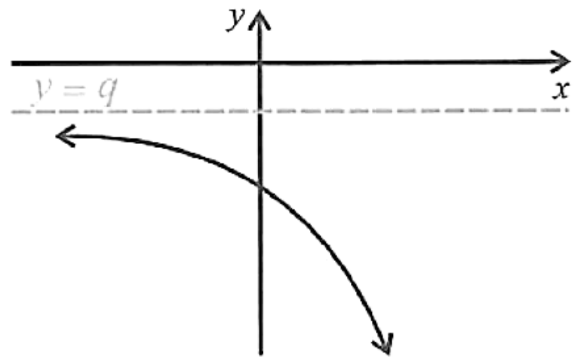
$a > 0$ and $b > 1$

The graph lies above the horizontal asymptote and is an increasing function.



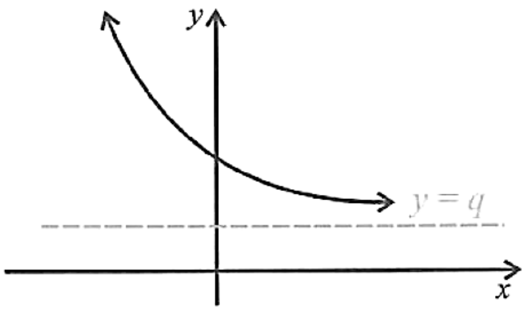
$a < 0$ and $b > 1$

The graph lies below the horizontal asymptote and is a decreasing function.



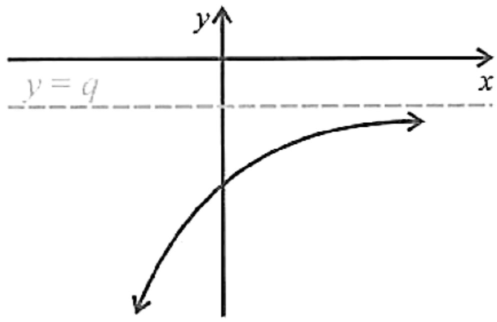
$a > 0$ and $0 < b < 1$

The graph lies above the horizontal asymptote and is a decreasing function.



$a < 0$ and $0 < b < 1$

The graph lies below the horizontal asymptote and is an increasing function.



(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

TRANSFORMATION OF FUNCTIONS

REFLECTIONS AND TRANSLATIONS

Given $f(x) = \frac{2}{x+1} - 3$	**Given $f(x) = 2.3^{x-2} + 4$**	**Given $f(x) = x^2 + 5x + 6$**
a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$. **Solution** $g(x) = f(x+3) + 2$ $= \frac{2}{x+3+1} - 3 + 2$ $= \frac{2}{x+4} - 1$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -\left(\frac{2}{x+1} - 3\right)$ $= -\frac{2}{x+1} + 3$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$.	a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$. **Solution** $g(x) = f(x+3) + 2$ $= 2.3^{x+3-2} + 4 + 2$ $= 2.3^{x+1} + 6$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -(2.3^{x-2} + 4)$ $= -2.3^{x-2} - 4$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$	a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units down and 3 units to right. Determine the equation of $g(x)$. **Solution** $g(x) = f(x-3) - 2$ $= (x-3)^2 + 5(x-3) + 6 - 2$ $= x^2 - 6x + 9 + 5x - 15 + 4$ $= x^2 - x - 2$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -(x^2 + 5x + 6)$ $= -x^2 - 5x - 6$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$.

Solution $m(x) = f(-x)$ $= \frac{2}{-x+1} - 3$ $= \frac{2}{-(x-1)} - 3$ $= -\frac{2}{x-1} - 3$	$= 2.3^{-x-2} + 4$ $= 2.3^{-(x+2)} + 4$ $= 2.\left(\frac{1}{3}\right)^{x+2} + 4$	**Solution** $m(x) = f(-x)$ $= (-x)^2 + 5(-x) + 6$ $= x^2 - 5x + 6$

<u>Given a function $f(x)$:</u>

(a) $f(x) + c$ means the graph is shifted by ''c'' units up. E.g. $g(x) = x^2 + 4$ means the graph of $f(x) = x^2$ is shifted by 4 units up.

(b) $f(x) - c$ means the graph is shifted by ''c'' units up. E.g. $g(x) = 2^x - 2$ means the graph of $f(x) = 2^x$ is shifted by 2 units up.

(c) $f(x + c)$ means the graph is shifted by ''c'' units to the left. E.g. $g(x) = \frac{2}{x+2}$ means the graph of $f(x) = \frac{2}{x}$ is shifted by 2 to the left.

(d) $f(x - c)$ means the graph is shifted by ''c'' units to the right. E.g. $g(x) = 2(x-5)^2$ means the graph of $f(x) = 2x^2$ is shifted by 5 units to the right.

(e) $f(-x)$ is the reflection of $f(x)$ about the y-axis. E.g. $g(x) = 2^{-x}$ is the reflection of $f(x) = 2^x$ about the y-axis.

(f) $-f(x)$ is the reflection of $f(x)$ about the x-axis. $g(x) = \frac{-2}{x+2} - 4$ is the reflection of $f(x) = \frac{2}{x+2} + 4$ about the x-axis.

With the transformations above, the effects of parameters on different functions discussed above can be easily understood

INVERSE FUNCTIONS

The concept of a function

A function f, is defined as a relationship between values, where each input value maps to one output value.
In other words, for an equation to be called a function, there can only be one y - value for a particular x-value.

There are two types of functions:
1. **One-to-One Functions**
2. **Many-to-One Functions**

ONE-TO-ONE FUNCTIONS

A one-to-one **function is a function where there is a single y-value for a particular x-value.**

MANY-TO-ONE FUNCTIONS

A function cannot have more than one y value to each x value. However, a function can have more than one x value for a particular y value. **These are known as** many to-one **functions.**

VERTICAL LINE TEST

To test if a graph is a function, use the vertical line **test.** If a vertical line (a line parallel to the y-axis) touches the graph more than once at any point, the graph is not a function. You don't have to draw a line, just hold a ruler parallel to the y-axis and move it along the graph. If the ruler touches the graph more than once for a single x value, anywhere on the graph, then the graph is not a function. In the case of the graph not being a function, it is said to be a relation.

HORIZONTAL LINE TEST

If a graph passes the vertical line test, it is a function. **The** horizontal line **test can be used to determine what type of function the graph represents.**

If a horizontal line (a line parallel to the x-axis) is drawn and moved along the graph and it **touches the graph more than once** at any point, it is a **many-to-one function** (many x-values to a single y-value). Otherwise, it is a one-to-one function.

(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics

- The inverse of a function takes the y-values (range) of the function to the corresponding x-values (domain) and vice versa. Therefore the x and y values are interchanged.
- The function is reflected along the line $y = x$ to form the inverse.
- The notation for the inverse of a function is f^{-1}.

Inverse Function

- N.B The domain of the inverse is the range of the function and the range of the inverse is the domain of the function.
- When the function is increasing, its inverse also increases. When the function decreases, its inverse will also decrease.

Inverse function : Linear ($y = ax + q$)

Example 1

Given $f(x) = 2x + 6$.

1. Determine $f^{-1}(x)$

2. Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axis

Solutions

1. In order to find the inverse of a function, there are two steps:

STEP 1: Swap the x and y

$$y = 2x + 6$$

becomes $x = 2y + 6$

We then rewrite the equation to make y the subject of the formula.
Therefore,

STEP 2: make y the subject of the formula

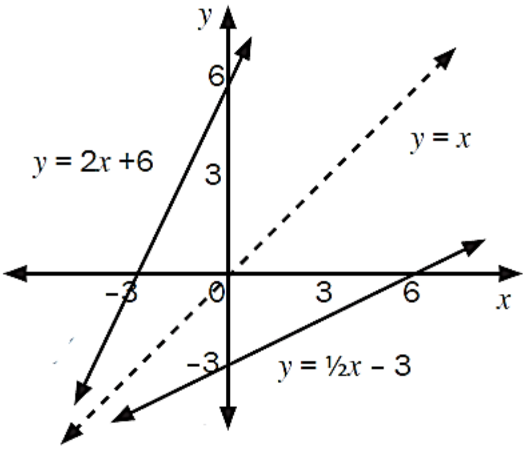
$x = 2y + 6$

$x - 6 = 2y$

So $y = \frac{1}{2}x - 3$

We can say that the inverse function $f^{-1}(x) = \frac{1}{2}x - 3$

2.



Inverse function : Quadratic ($y = ax^2$)

Example 2

a) Sketch $f(x) = 2x^2$

b) Determine the inverse of $f(x)$

c) Sketch $f^{-1}(x)$ and $y = x$ on the same axes as $f(x)$

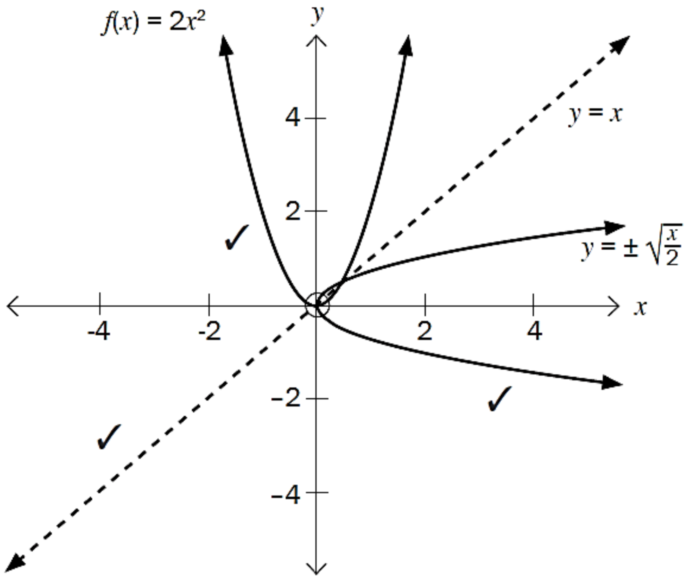
a) & c)

Solution

1. b) $y = 2x^2$

$x = 2y^2$ ✓

$y = \pm\sqrt{\frac{x}{2}}$ ✓



- This is not a function.
- Check it with a vertical line test. There are two y-values for one x-value.
- Not all inverses of functions are also functions. Some inverses of functions are relations.
- If an inverse is not a function, then we can restrict the **domain** of the **function** in order for the inverse to be a function.

- To make the inverse a function, we need to choose a set of x-values in the function and work only with those. We call this **'restricting the domain'**.
- A one to one function has an inverse that is a function

  Example: $y = 3x + 4$ is a one to one function. For every x value there is one and only one y value

- A many to one function has an inverse that is not a function. However, we can restrict the domain of the function to make its inverse a function.

  Example: $y = 2x^2$ is a many to one function. For two or many x values there is one y value. (if $x = 2$, then $y = 8$.
  If $x = -2$, then $y = 8$). Therefore, its inverse$y = \pm\sqrt{\frac{x}{2}}$, is not a function.
- To check for a function, draw a vertical line. If any vertical line cuts the graph in only one place, the graph is a function.
  If any vertical line cuts the graph in more than one place, then the graph is not a function.
- To check for a one-to-one function, draw a horizontal line. If any horizontal line cuts the graph in only one place, the graph is a one-to-one function. If any horizontal line cuts the graph in more than one place, then the graph is a many-to-one function.

Example 3

Given: $g(x) = -x^2$ where $x \leq 0$

(a) Write down the inverse of g, g^{-1} in the form $h(x) =$ ....

(b) Sketch the graphs of g, h and $y = x$ on the same set of axis.

Solutions

(a) $y = -x^2$

$x = -y^2$

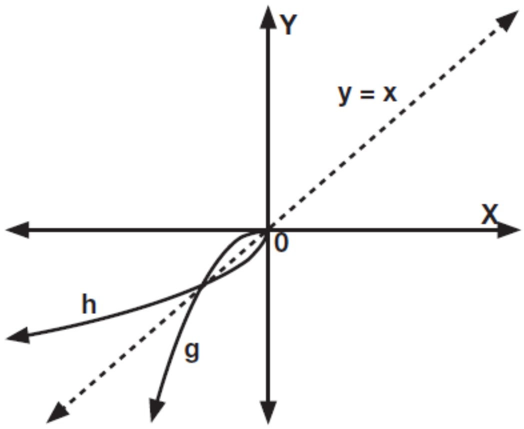
$-x = y^2$

$\pm\sqrt{-x} = y$

$-\sqrt{-x} = y$ where $x \leq 0$

$\therefore h(x) = -\sqrt{-x}$

(b)



Inverse function : Exponential $y = b^x; (b > 0, b \neq 1)$

Example 4

Given: $f(x) = 2^x$

a) Determine f^{-1} in the form $y = \ldots\ldots$

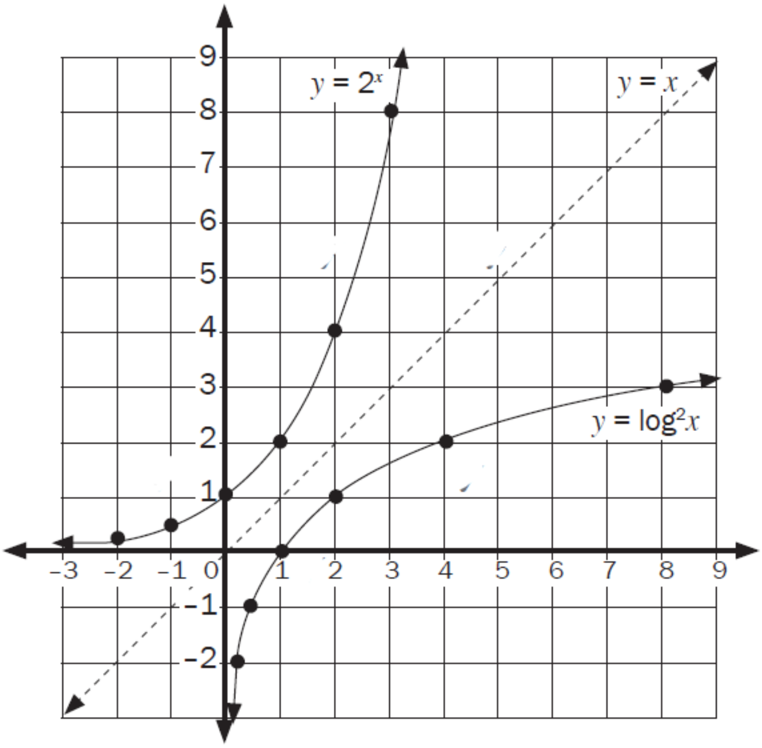
b) Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axes.

c) Write the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) The inverse of the exponential function $y = 2^x$ is $x = 2^y$ which can be written as $y = \log_2 x$.

b)



c) The domain and Range of $f(x)$

Domain: $x \in R$

Range: $y > 0$

The domain and Range of $f^{-1}(x)$

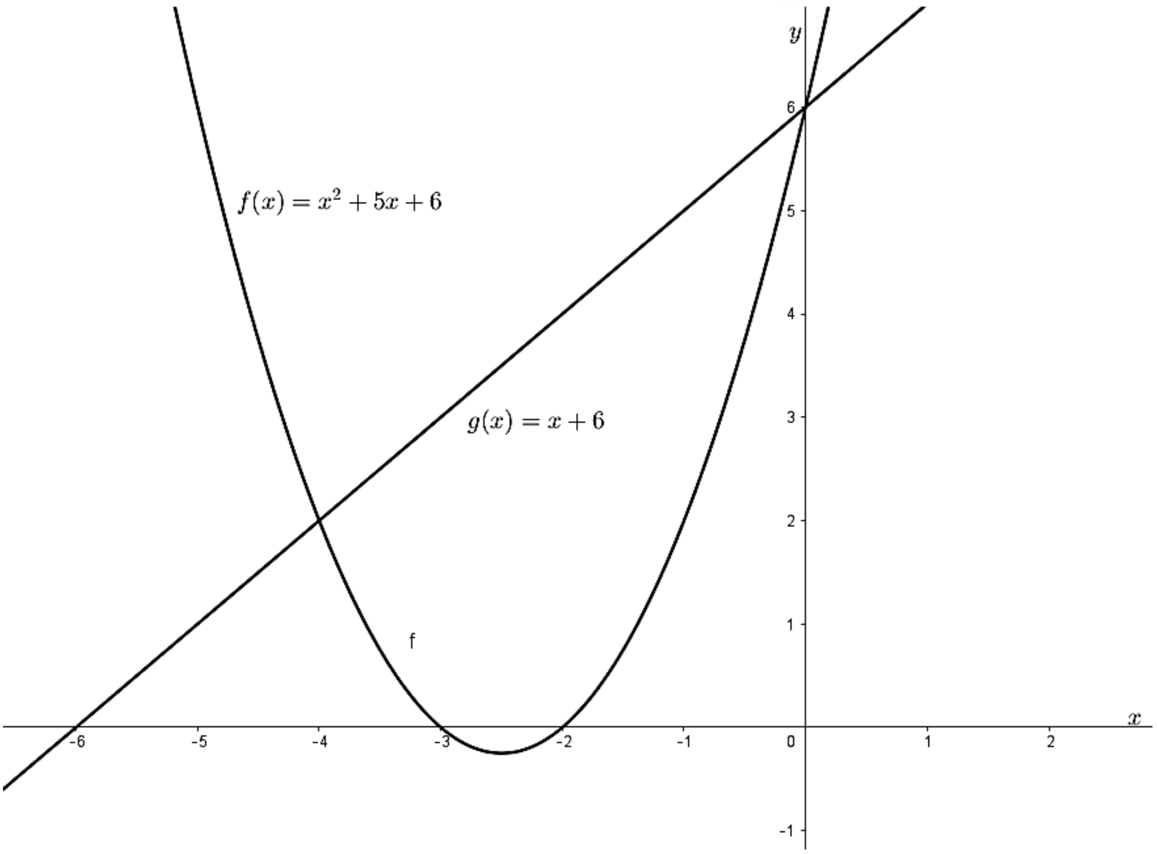
Domain: $x > 0$

Range: $y \in 0$

<u>COMBINATIONS</u>

Take note of the following when working with combination of functions:

Consider the graphs of f and g below



For $f(x) > 0$ or $f(x) < 0$	For $f(x) > g(x)$ or $f(x) < g(x)$	For $f(x).g(x) > 0$ or $f(x).g(x) < 0$
Focus on the $x - axis$. $f(x) > 0$ means where the graph of $f(x)$ is positive, which will be above the $x - axis$. And $f(x) < 0$ means where the graph of $f(x)$ is negative, which will be below the $x - axis$.	$f(x) > g(x)$ means where the graph of $f(x)$ is above the graph of $g(x)$. And $f(x) < g(x)$ means where the graph of $f(x)$ is below the graph of $g(x)$.	**$f(x).g(x) > 0$ means where the product of $f(x)$ and $g(x)$ is positive.** **And $f(x).g(x) < 0$ means where the product of $f(x)$ and $g(x)$ is negative.**

THE AVERAGE GRADIENT BETWEEN TWO POINTS

The **average gradient** of a function between any two points is defined to be the **gradient of the line** joining the two points.

Average gradient = $\frac{y_2 - y_1}{x_2 - x_1}$

PAPER A

QUESTION 4

Given: $g(x)=\dfrac{1}{x-1}+2$

4.1 Write down the equations of the asymptotes of g. (2)

4.2 Draw a graph of g, indicating any intercepts with the axes and asymptotes. (4)

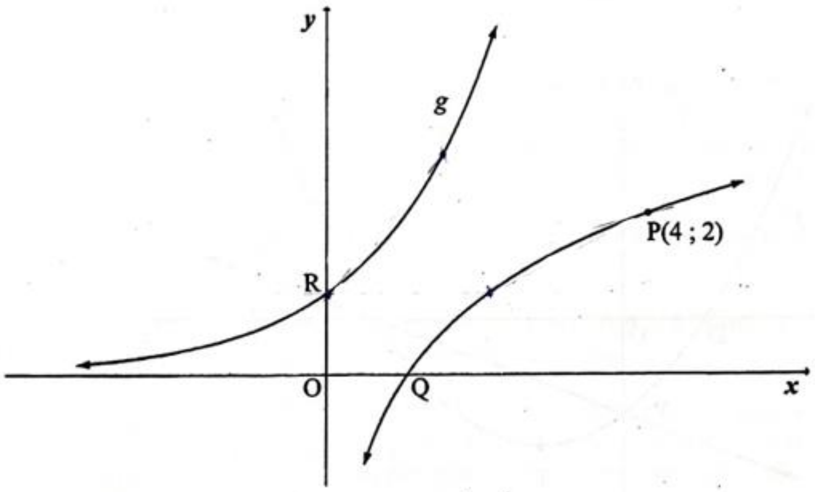
4.3 Determine the values of x where $g(x)>0$. (2)

4.4 Determine the equation of the axis of symmetry of g which has a negative gradient. (2)

[10]

QUESTION 5

In the diagram, the graphs of $f(x)=\log_a x$ and g are drawn. Graph g is the reflection of f in the line $y=x$. Graph f passes through the point P(4 ; 2). Q is the x-intercept of f and R is the y-intercept of g.



5.1 Write down the coordinates of P$'$, the image of P on g. (2)

5.2 Show that $a=2$. (2)

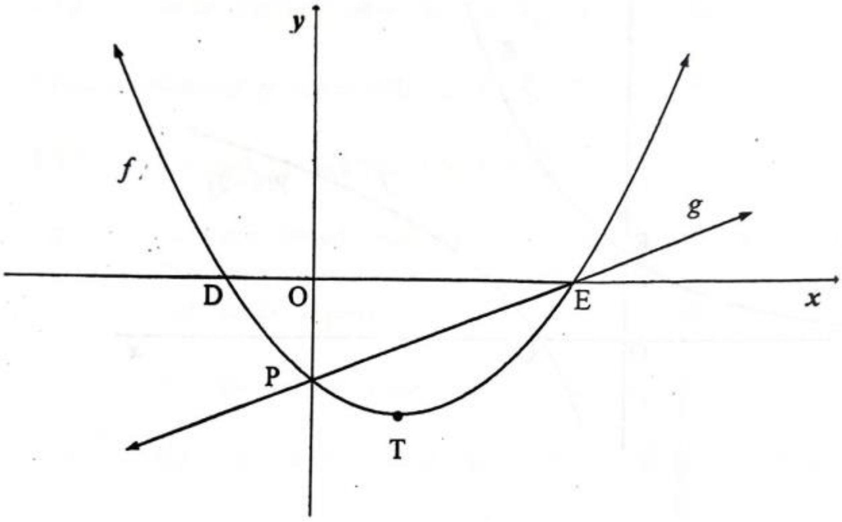
5.3 Write down the equation of g in the form $y=\ldots$ (1)

5.4 T is a point on f in the first quadrant where TR is parallel to the x-axis. Calculate the area of ΔRTP$'$. (4)

[9]

QUESTION 6

The graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = mx + c$ are drawn below. D and E are the x-intercepts and P is the y-intercept of f. The turning point of f is T(1 ; – 4). The graphs of f and g intersect at P and E.



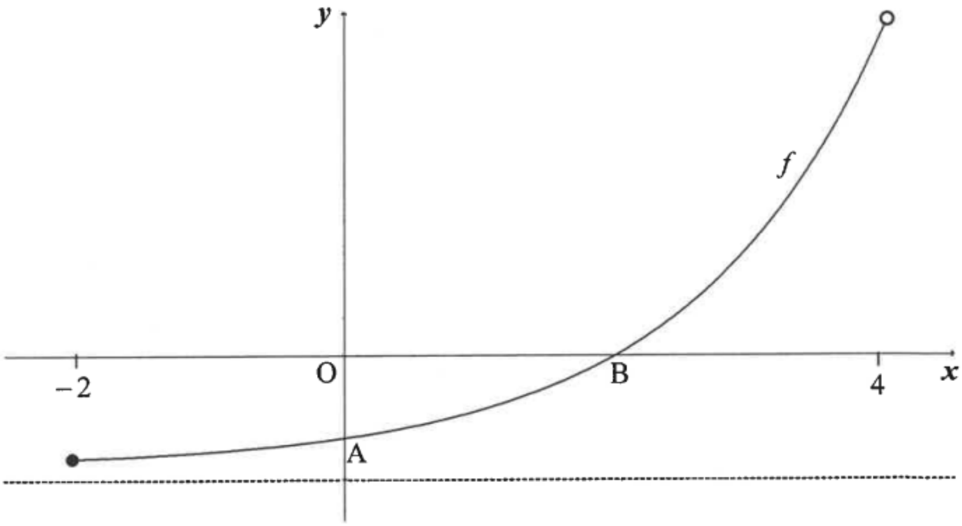
6.1	Write down the range of f.	(1)
6.2	Calculate the coordinates of D and E.	(3)
6.3	Determine the equation of g.	(2)
6.4	Write down the values of x for which $f(x) - g(x) > 0$.	(2)
6.5	Determine the maximum vertical distance between h and g if $h(x) = -f(x)$ for $x \in [-2 ; 3]$.	(5)
6.6	Given: $k(x) = g(x) - n$. Determine n if k is a tangent to f.	(5)
		[18]

PAPER B

QUESTION 4

Sketched below is the graph of $f(x) = 2^x - 4$ for $x \in [-2\,;4)$.

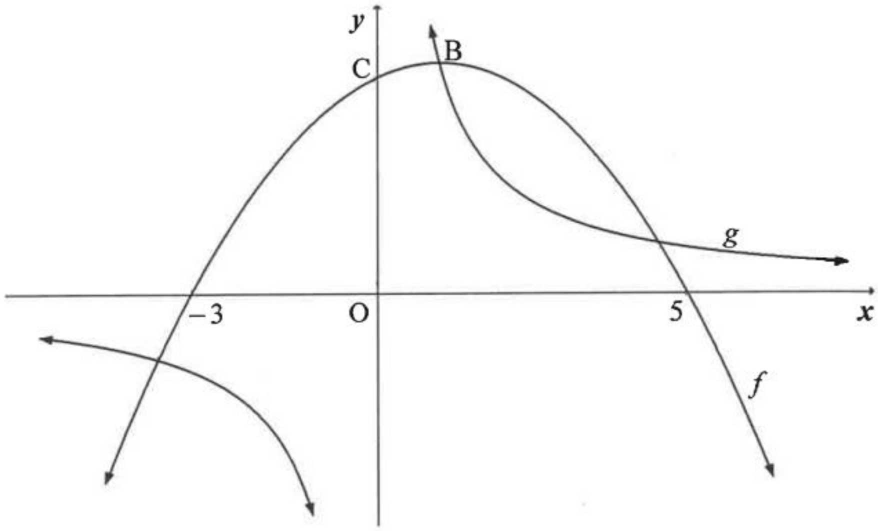
A and B are respectively the y- and x-intercepts of f.



4.1	Write down the equation of the asymptote of f.	(1)
4.2	Determine the coordinates of B.	(2)
4.3	Determine the equation of k, a straight line passing through A and B in the form $k(x) = \ldots$	(3)
4.4	Calculate the vertical distance between k and f at $x = 1$	(3)
4.5	Write down the equation of g if it is given that $g(x) = f(x) + 4$	(1)
4.6	Write down the domain of g^{-1}.	(2)
4.7	Write down the equation of g^{-1} in the form $y = \ldots$	(2)
		[14]

QUESTION 5

The graphs of $f(x)=-\frac{1}{2}(x-1)^2+8$ and $g(x)=\frac{d}{x}$ are drawn below. A point of intersection of f and g is B, the turning point of f. The graph f has x-intercepts at $(-3\,;0)$ and $(5\,;0)$ and a y-intercept at C.

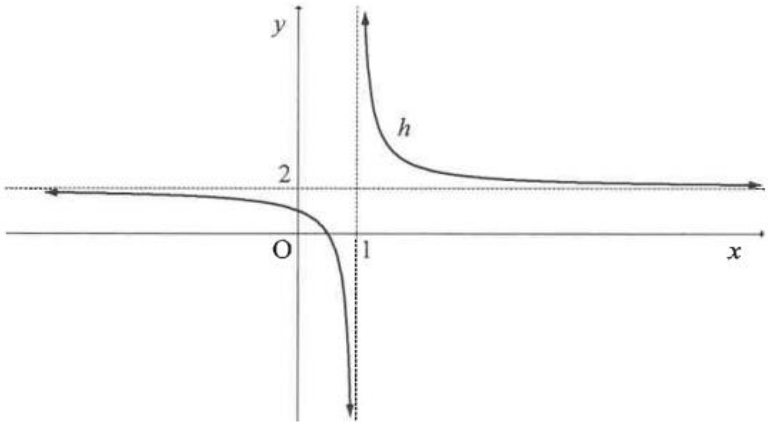


5.1	Write down the coordinates of the turning point of f.	(2)
5.2	Calculate the coordinates of C.	(2)
5.3	Calculate the value of d.	(1)
5.4	Write down the range of g.	(1)
5.5	For which values of x will $f(x).g(x)\leq 0$?	(3)
5.6	Calculate the values of k so that $h(x)=-2x+k$ will not intersect the graph of g.	(5)
5.7	h is a tangent to g at R, a point in the first quadrant. Calculate t such that $y=f(x)+t$ intersects g at R.	(4)
		[18]

PAPER C

QUESTION 4

4.1 Sketched below is the graph of $h(x)=\frac{1}{x+p}+q$. The asymptotes of h intersect at $(1\,;2)$.



4.1.1 Write down the values of p and q. (2)

4.1.2 Calculate the coordinates of the x-intercept of h. (2)

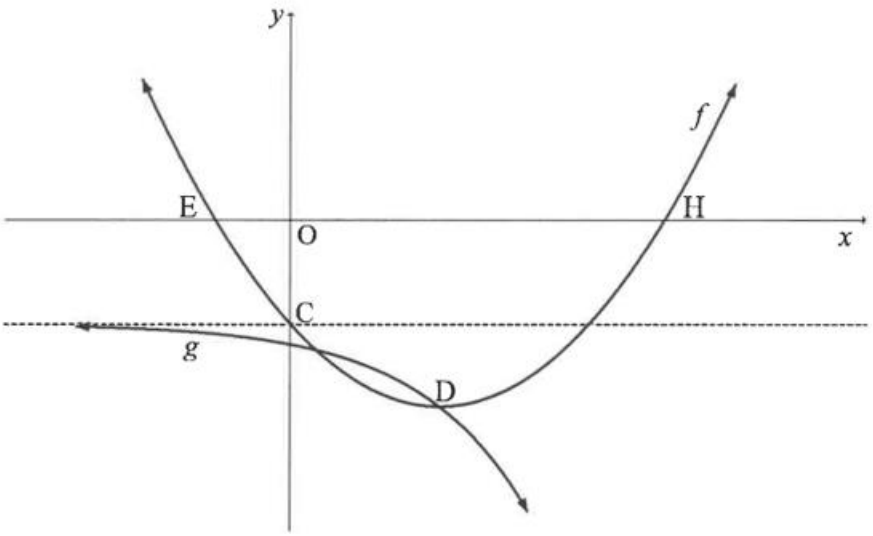
4.1.3 Write down the x-coordinate of the x-intercept of g if $g(x)=h(x+3)$. (2)

4.1.4 The equation of an axis of symmetry of h is $y=x+t$. Determine the value of t. (2)

4.1.5 Determine the values of x for which $-2\le\frac{1}{x-1}$. (3)

4.2 The graphs of $f(x) = x^2 - 4x - 5$ and $g(x) = a.2^x + q$ are sketched below.

- E and H are the x-intercepts of f.
- C is the y-intercept of f and lies on the asymptote of g.
- The two graphs intersect at D, the turning point of f.

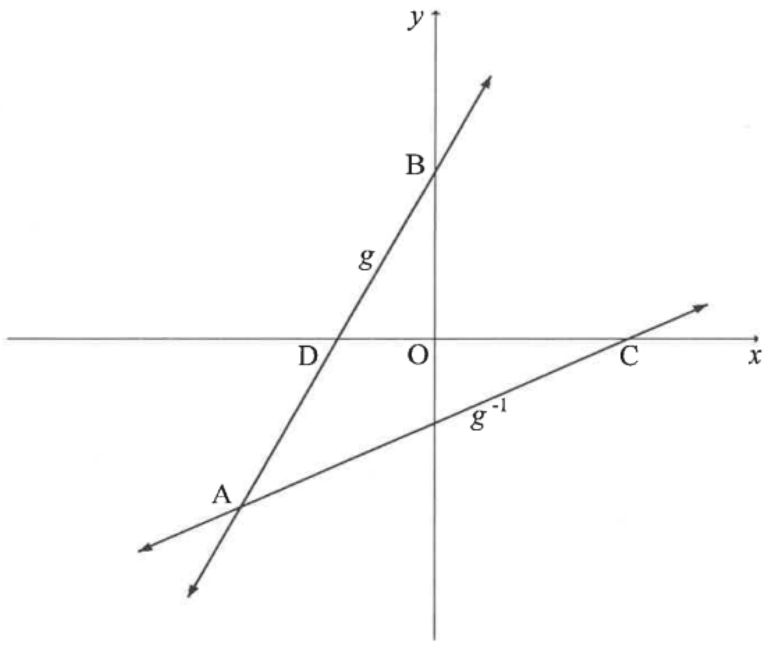


4.2.1	Write down the y-coordinate of C.	(1)
4.2.2	Determine the coordinates of D.	(2)
4.2.3	Determine the values of a and q.	(3)
4.2.4	Write down the range of g.	(1)
4.2.5	Determine the values of k for which the value of $f(x) - k$ will always be positive.	(2)
		[20]

QUESTION 5

The graphs of $g(x) = 2x + 6$ and g^{-1}, the inverse of g, are shown in the diagram below.

- D and B are the x- and y-intercepts respectively of g.
- C is the x-intercept of g^{-1}.
- The graphs of g and g^{-1} intersect at A.



5.1 Write down the y-coordinate of B. (1)

5.2 Determine the equation of g^{-1} in the form $g^{-1}(x) = mx + n$. (2)

5.3 Determine the coordinates of A. (3)

5.4 Calculate the length of AB. (2)

5.5 Calculate the area of ΔABC. (5)

[13]

PAPER D

QUESTION 5

Given: $f(x) = -2x^2 + x + 6$

5.1 Calculate the coordinates of the turning point of f. (4)

5.2 Determine the y-intercept of f. (1)

5.3 Determine the x-intercepts of f. (4)

5.4 Sketch the graph of f showing clearly all intercepts with the axes and turning point. (3)

5.5 Determine the values of k such that $f(x) = k$ has equal roots. (2)

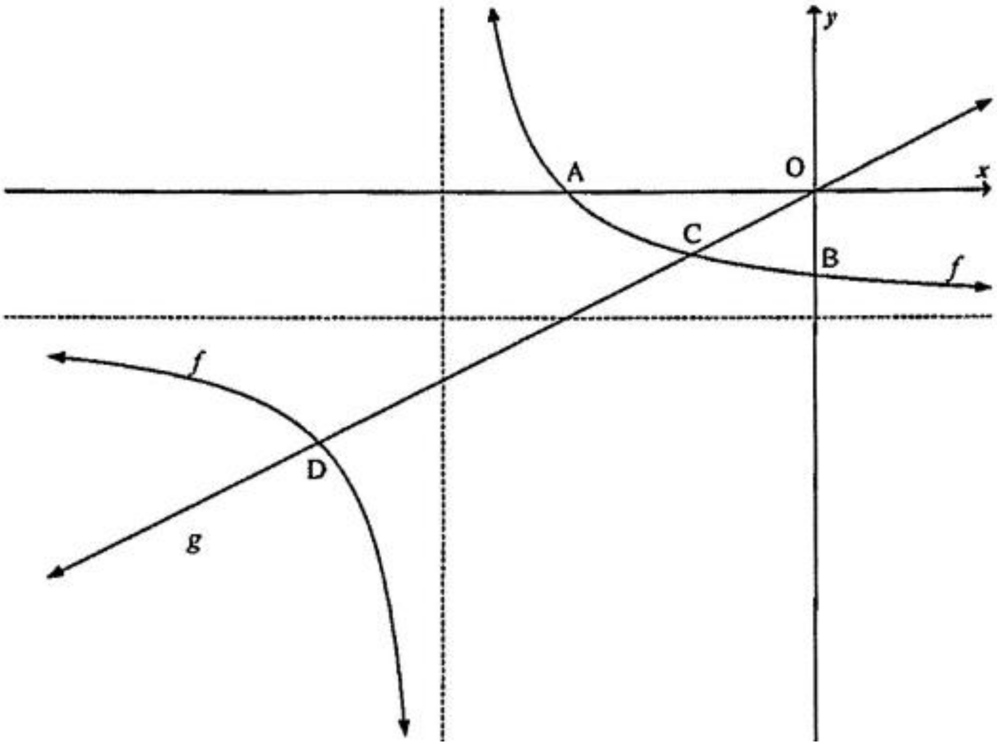
5.6 If the graph of f is shifted two units to the right and one unit upwards to form h, determine the equation h in the form $y = a(x+p)^2 + q$. (3)

QUESTION 6

The diagram below shows the graph of $f(x) = \dfrac{1}{x+3} - 1$ and $g(x) = \dfrac{1}{2}x$.

The graph of f intersects the x-axis at A and the y-axis at B.

The graph of f and g intersect at points C and D.



6.1 Write down the equations of the asymptotes of f. (2)

6.2 Determine the domain of f. (2)

6.3 Calculate the length of:

6.3.1 OB (2)

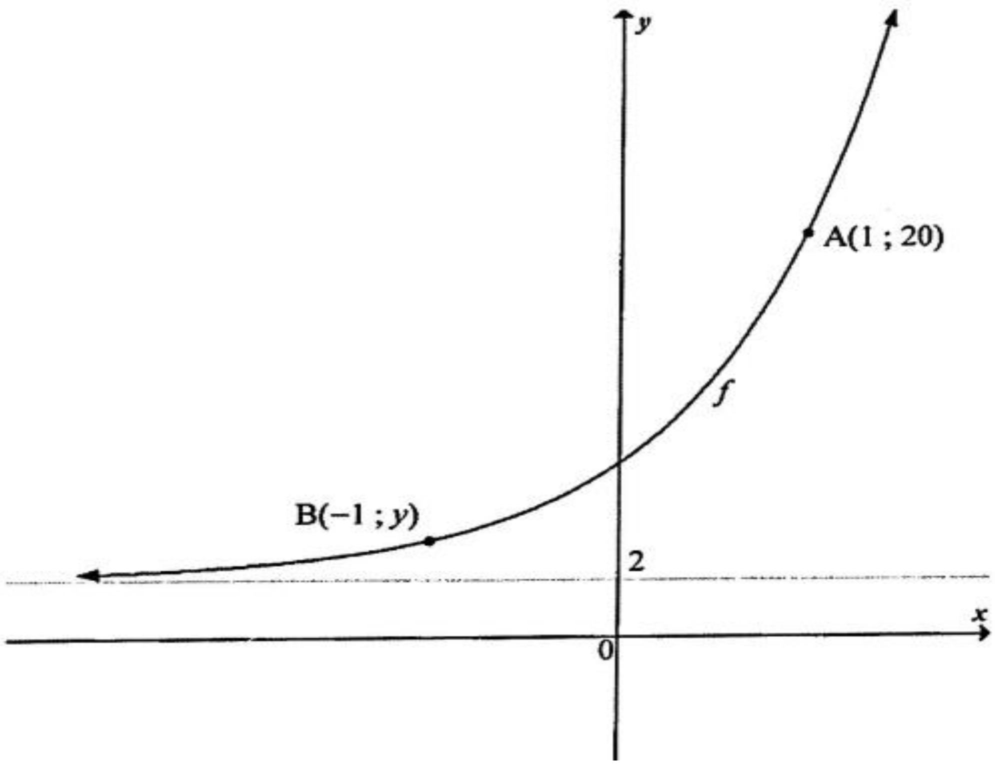
6.3.2 OA (3)

6.4 Determine the coordinates of C and D. (6)

6.5 Use the graphs to obtain the solution to: $\frac{1}{x+3} \geq \frac{x+2}{2}$ (4)

QUESTION 7

The sketch below is the graph of $f(x) = 2.b^{x+1} + q$.
The graph of f passes through the points A(1 ; 20) and B (–1 ; y).
The line $y = 2$ is an asymptote of f.



7.1 Show that the equation of f is $f(x)=2(3)^{x+1}+2$ (3)

7.2 Calculate the y-coordinate of the point B. (1)

7.3 Determine the average gradient of the curve between the points A and B. (2)

7.4 A new function h is obtained when f is reflected about its asymptote. Determine the equation of h. (2)

7.5 Write down the range of h. (1)

PAPER E

QUESTION 5

Given $f(x)=\dfrac{-4}{2-x}-1$

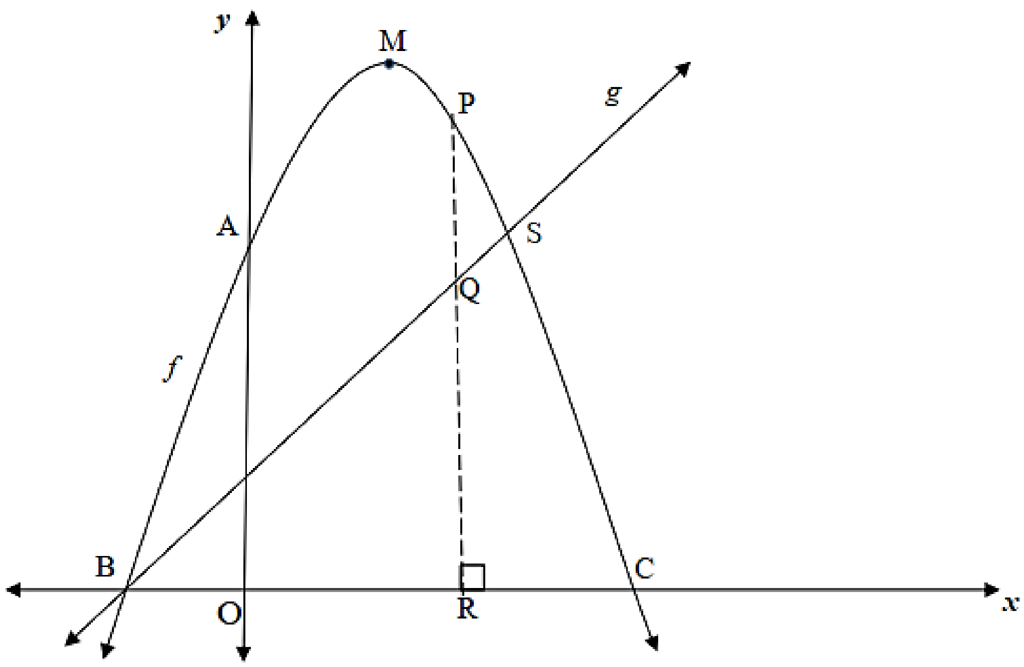
5.1 Write down the equations of the vertical and horizontal asymptotes of f. (2)

5.2 Determine the intercepts of the graph of f with the axes. (3)

5.3 Draw the graph of f. Show all intercepts with the axes as well as the asymptotes of the graph. (4)

QUESTION 6

In the diagram, the graphs of $f(x)=-x^2+5x+6$ and $g(x)=x+1$ are drawn below.
The graph of f intersects the $x-$axis at B and C and the $y-$axis at A.
The graph of g intersects the graph of f at B and S. PQR is perpendicular to the $x-$axis with points P and Q on f and g respectively. M is the turning point of f.



6.1 Write down the co-ordinates of A. (1)

6.2 S is the reflection of A about the axis of symmetry of *f*. Calculate the coordinates of S. (2)

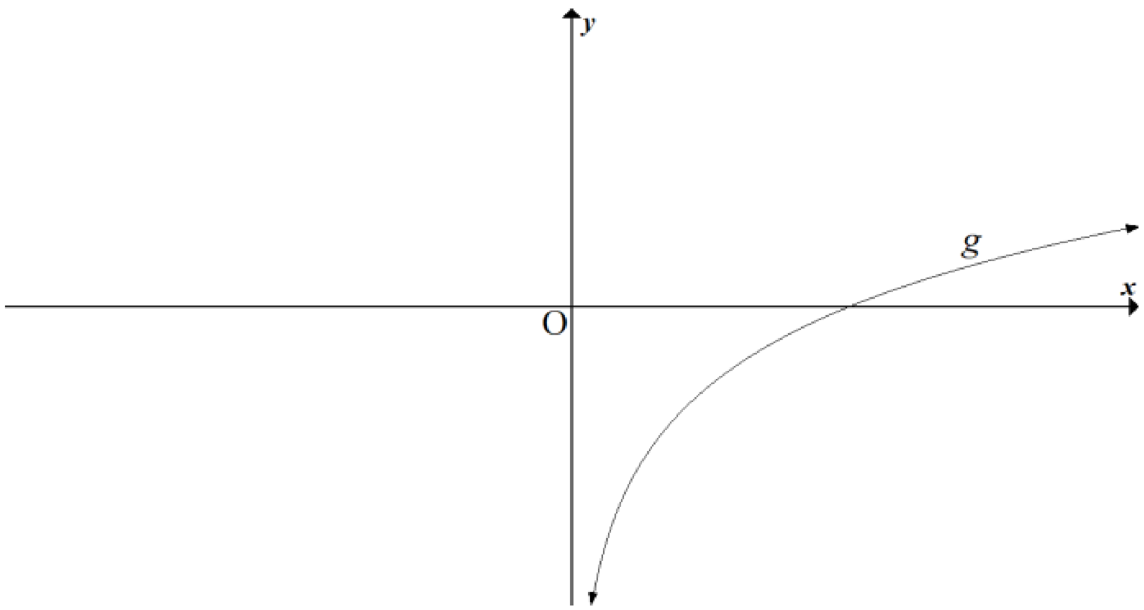
6.3 Calculate the coordinates of B and C. (3)

6.4 If PQ = 5 units, calculate the length of OR. (4)

6.5 Calculate the:
6.5.1 Coordinates of M. (4)
6.5.2 Maximum length of PQ between B and S. (4)

QUESTION 7

In the diagram, the graph of $g(x) = \log_5 x$ is drawn.



7.1 Write down the equation of g^{-1}, the inverse of g, in the form $y = \ldots$ (2)

7.2 Write down the range of g^{-1}. (1)

7.3 Calculate the value(s) of x for which $g(x) \leq -4$. (4)

PAPER F

5.2 Given: $h(x) = 4(2^{-x}) + 1$

5.2.1 Determine the coordinates of the y-intercept of h. (2)

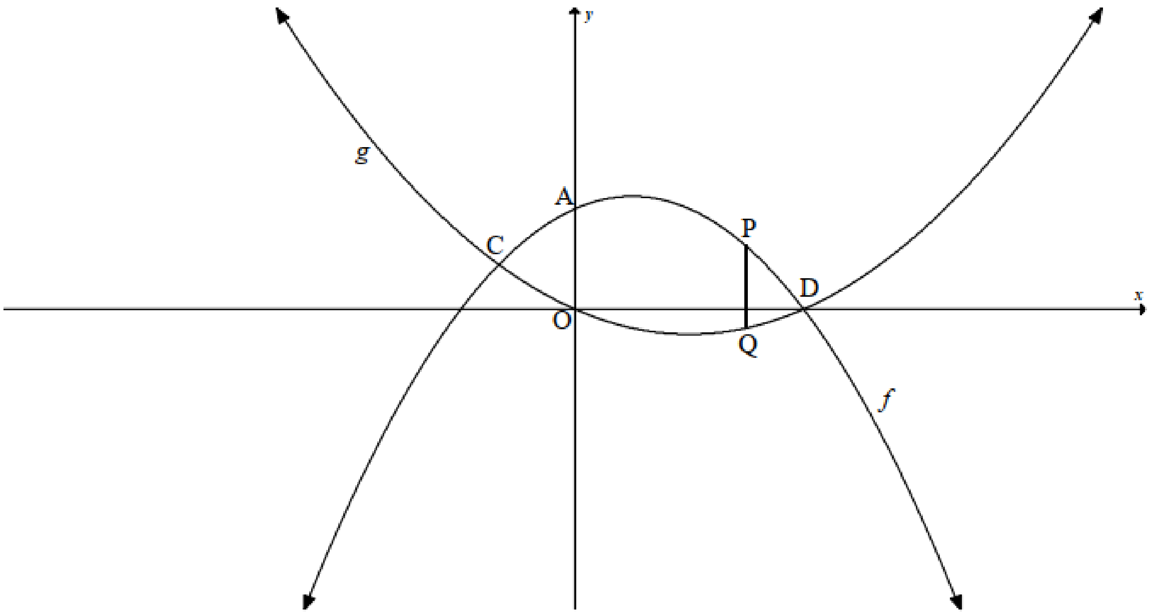
5.2.2 Explain why h does not have an x-intercept. (2)

5.2.3 Draw a sketch graph of h, clearly showing all asymptotes, intercepts with the axes and at least one other point on h. (3)

5.2.4 Describe the transformation from h to g if $g(x) = 4(2^{-x} + 2)$. (2)

QUESTION 6

In the diagram, the graphs of $f(x) = -x^2 + x + 2$ and $g(x) = \frac{1}{2}x^2 - x$ are drawn below. f and g intersect at C and D. A is the $y-$intercept of f. P and Q are any points on f and g respectively. PQ is parallel to the $y-$axis.



6.1 Write down the co-ordinates of A. (1)

6.2 Calculate the coordinates of C and D. (5)

6.3 Determine the values of x for which $f(x) \leq g(x)$. (2)

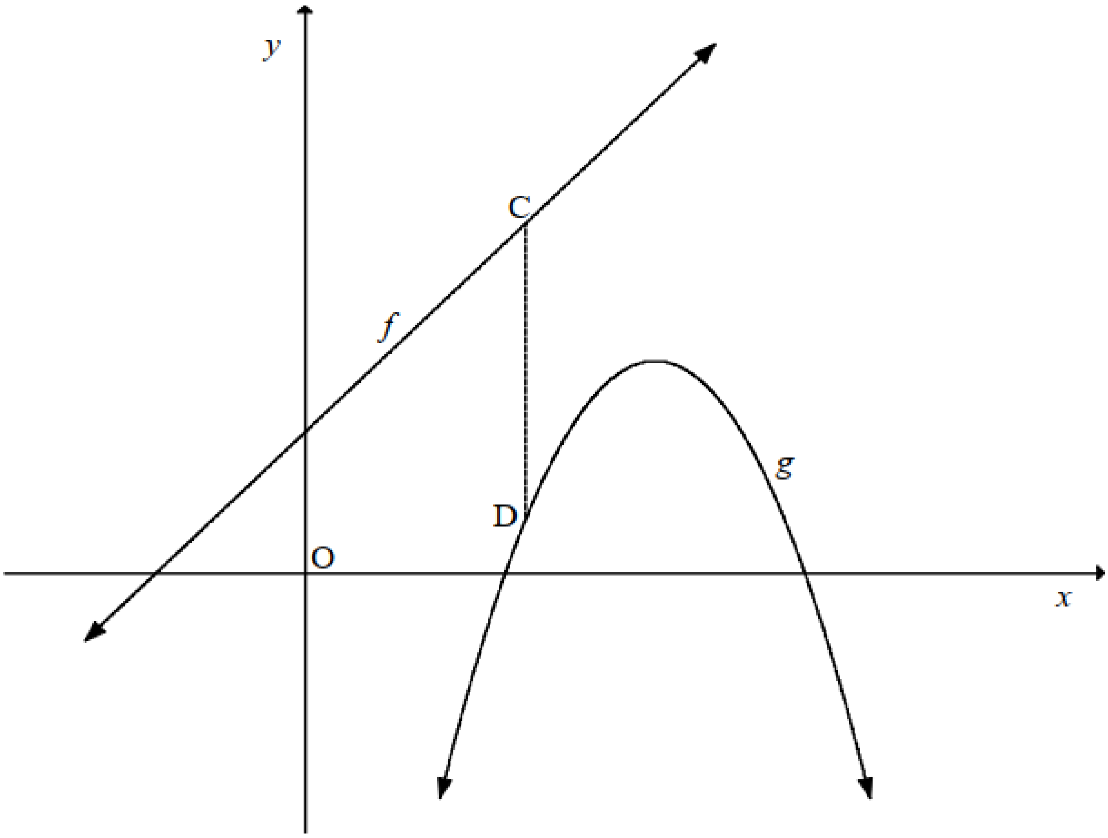
6.4 Calculate the maximum length of PQ where line PQ is between C and D. (4)

6.5 Calculate the value of x where the gradient of f is equal to 3. (3)

6.6 Determine the values of k for which $f(x) = k$ has two positive unequal roots. (4)

QUESTION 7

The sketch below shows the graphs of $f(x) = 2x + 3$ and $g(x) = -2x^2 + 14x + k$.
C is any point on f and D any point on g, such that CD is parallel to the y-axis.
k is a value such that C lies above D.



7.1 Write down a simplified expression for the length of CD in terms of x and k. (3)

7.2 If the minimum length of CD is 5, calculate the value of k. (4)

PAPER G

QUESTION 5

5.1 Consider the function $f(x) = \left(\frac{5}{6}\right)^x$

5.1.1 Write down the equation of h, the reflection of f in the y-axis. (1)

5.1.2 Write down the equation of $f^{-1}(x)$ in the form $y = ...$ (2)

5.1.3 For which value(s) of x will $f^{-1}(x) \geq 0$? (2)

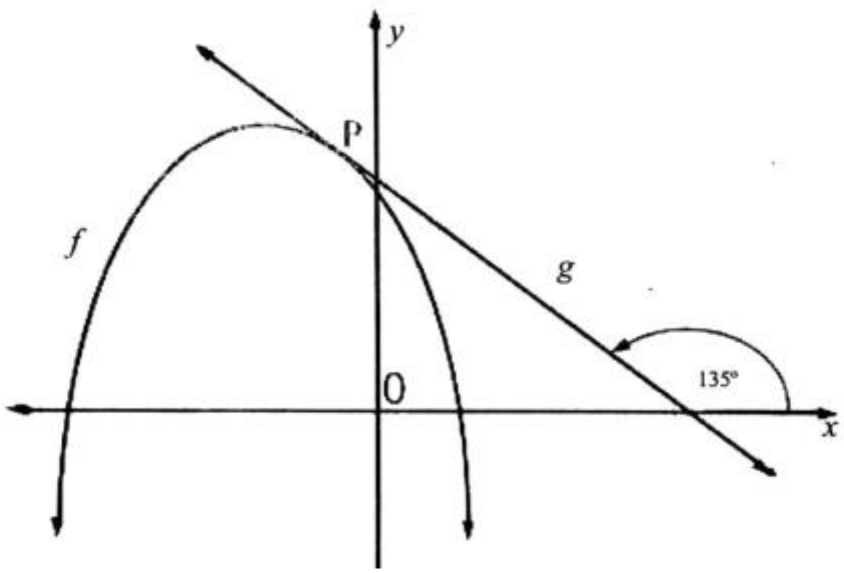
5.2 The function defined as $f(x) = ax^2 + bx + c$ has the following properties:

- $f'(-2,5) = 0$
- $f(1) = 0$
- $b^2 - 4ac > 0$
- $f(-2,5) = 6$

Draw a neat sketch graph of f. Clearly show all x-intercepts and turning point. (4)

QUESTION 6

The graphs of $f(x) = -2x^2 - 5x + 3$ and $g(x) = ax + q$ are sketched below.
The angle of inclination of g is 135°. Graph g is a tangent to f at point P.



6.1 Calculate the coordinates of the turning point of f. (3)

6.2 Write down the range of f. (1)

6.3 Calculate the coordinates of point P, the point of contact of f and g. (4)

6.4 Determine the value(s) of k for which the straight line $y = k$ is NOT a tangent to $y = 2x^2 + 5x - 3$. (2)

QUESTION 7

Given $f(x) = a^x$, where $a > 0$, passing through the point $(2\ ;\ \frac{1}{4})$ and $g(x) = 4x^2$.

7.1 Prove that $a = \frac{1}{2}$. (2)

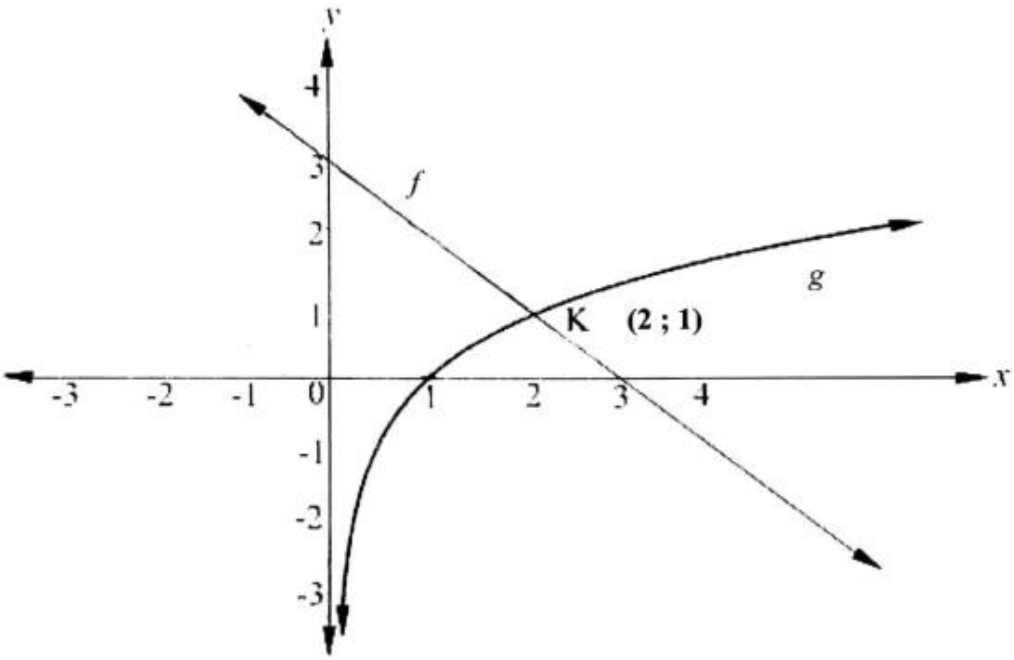
7.2 Determine the equation of $y = f^{-1}(x)$ in the form $y = \ldots$ (2)

7.3 Determine the equation of $y = h(x)$ where h is the reflection of f in the x-axis. (1)

7.4 How must the domain of $g(x)$ be restricted so that $g^{-1}(x)$ will be a function? (2)

QUESTION 8

The graphs of $f(x) = -x + 3$ and $g(x) = \log_2 x$ are drawn below.
Graphs f and g intersect at point $K(2\ ;\ 1)$.



8.1 Write down value(s) of x for which:

8.1.1 $f(x)-g(x)>0$ (2)

8.1.2 $g(x)\,.\,g^{-1}(x)\leq 0$ (2)

8.2 8.2.1 Write down the equation of g^{-1} in the form $y=\ldots$ (2)

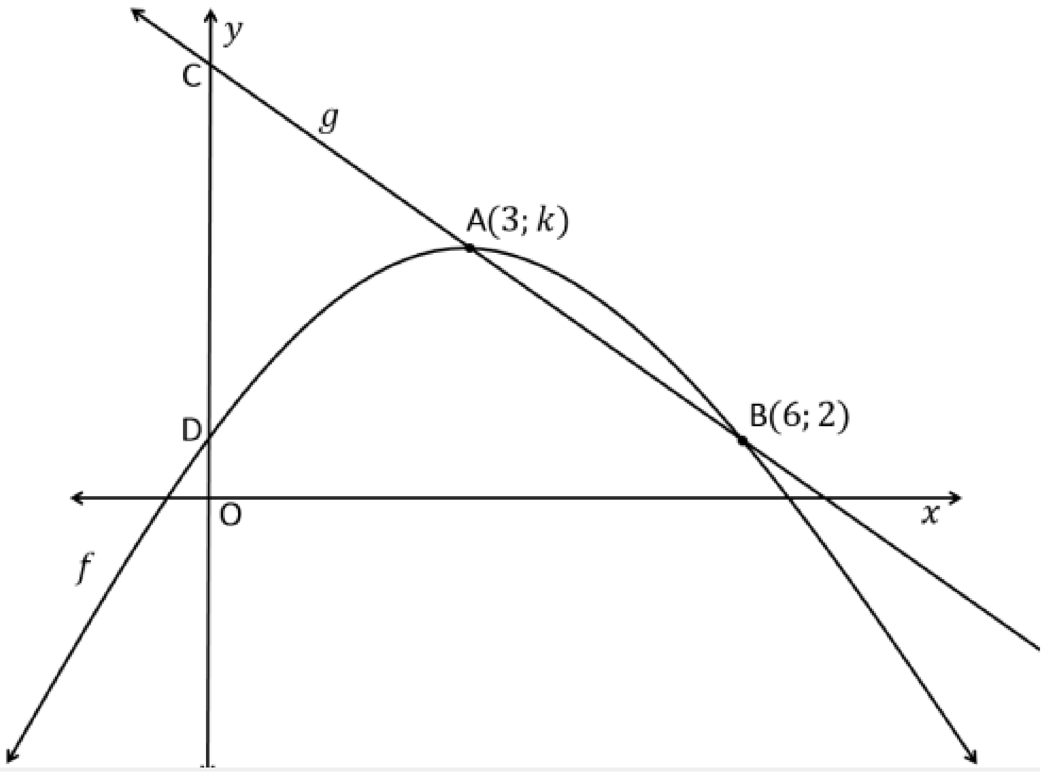
8.2.2 Explain how you could use the given sketch to solve the equation $\log_2(3-x)=x$. (2)

8.2.3 Write down the solution to $\log_2(3-x)=x$. (1)

PAPER H

QUESTION 5

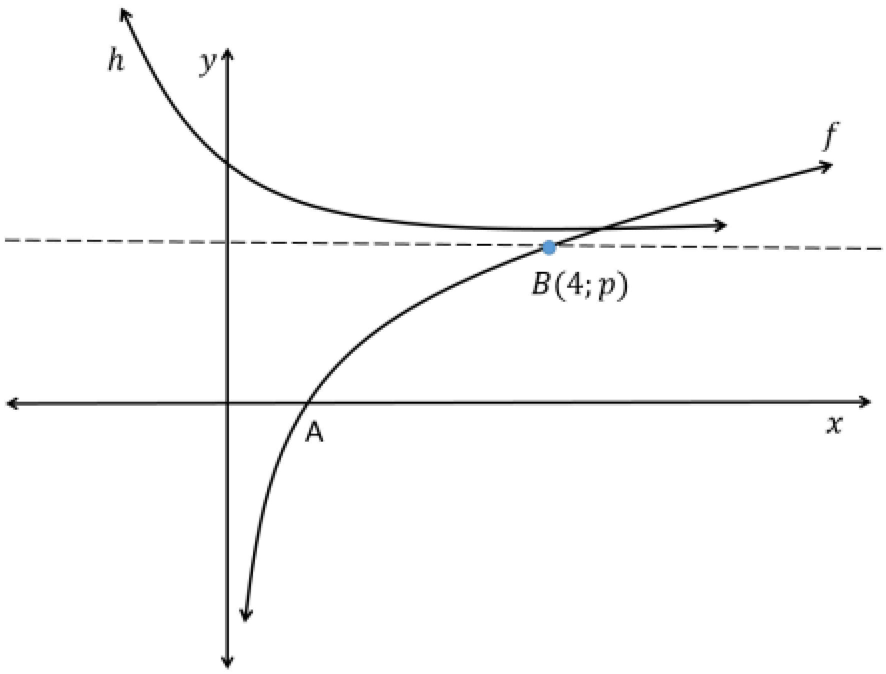
Sketched below are the graphs of $g(x) = -3x + 20$ and $f(x) = ax^2 + bx + c$.
Graph f has a turning point at $A(3; k)$. Graph f and g intersects at A and $B(6; 2)$.



5.1	Calculate the numerical value of k, the y-coordinate of A.	(2)
5.2	Determine the range of $y = -f(x)$.	(2)
5.3	Calculate the numerical values of a, b and c.	(6)
5.4	Determine the value(s) of x for which $f(x) > g(x)$.	(2)
5.5	Describe the nature of the roots for $f(x) - 11$.	(2)
5.6	Determine the value(s) of x for which $f'(x) \, . \, g'(x) > 0$.	(2)

QUESTION 6

Sketched below are the graphs of $h(x) = \left(\frac{1}{2}\right)^x + q$ and $f(x) = \log_2 x$.
Graph f and the asymptote of h intersect at $B(4; p)$.



6.1	Write down the coordinates of A, the x –intercept of f.	(1)
6.2	Determine the domain of f.	(1)
6.3	Determine the equation of f^{-1} in the form $y = \ldots$	(2)
6.4	Sketch the graph of f^{-1}. Clearly labelling the intercept(s) with the axes as well as the coordinates of any one other point on the graph.	(3)
6.5	Determine the equation of the asymptote of h.	(2)
6.6	Describe, in words, the transformation of h to f^{-1}.	(2)

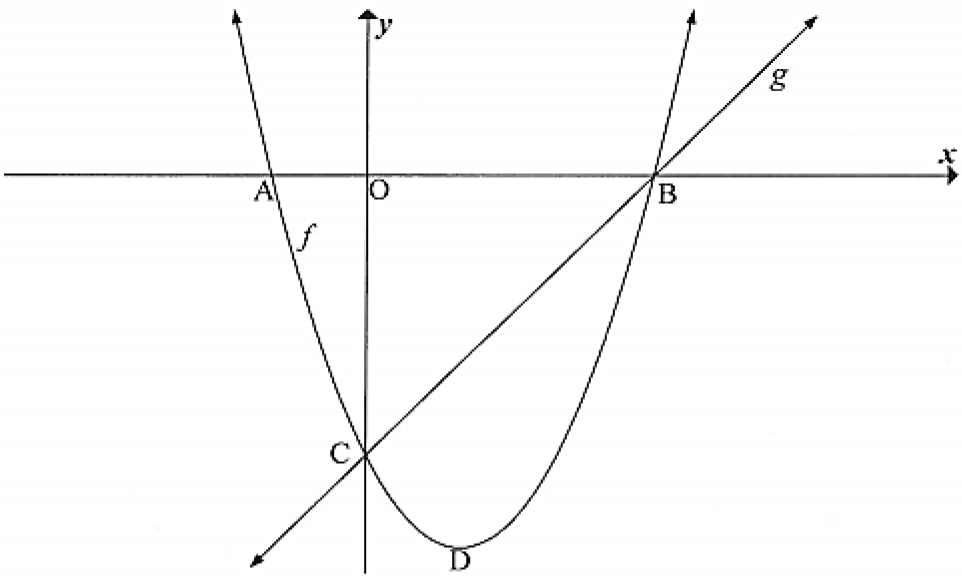
PAPER I

QUESTION 5

5.1 The sketch below shows the graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = x - 3$.

- A and B are the x-intercepts of f.
- The graphs of f and g intersect at C and B.

D is the turning point of f.



5.1.1	Determine the coordinates of C.	(1)
5.1.2	Calculate the length of AB.	(4)
5.1.3	Determine the coordinates of D.	(2)
5.1.4	Calculate the average gradient of f between C and D.	(2)
5.1.5	Calculate the size of $O\hat{C}B$	(2)
5.1.6	Determine the values of k for which $f(x) = k$ will have two unequal positive real roots.	(3)
5.1.7	For which values of x will $f'(x) \, . \, f''(x) > 0$?	(3)

5.2 The graph of a parabola f has x-intercepts at $x = 1$ and $x = 5$. $g(x) = 4$ is a tangent to f at P, the turning point of f. Sketch the graph of f, clearly showing the intercepts with the axes and the coordinates of the turning point. (5)

QUESTION 6

Given: $f(x)=\frac{1}{4}x^2$, $x\le 0$

6.1 Determine the equation of f^{-1} in the form $f^{-1}(x)=\ldots$ (3)

6.2 On the same system of axes, sketch the graphs of f and f^{-1}. Indicate clearly the intercepts with the axes, as well as another point on the graph of each of f and f^{-1}. (3)

6.3 Is f^{-1} a function? Give a reason for your answer. (2)

FINANCE, GROWTH AND DECAY

GRADE 11

Once you bought expensive items they will lose value over a period of time. E.g. car or furniture loses its value as it becomes a scrap over a period of time as it being used. In maths we called this is **DEPRECIATION.**

Other items gain value over a period of time.

E.g Property (house), house itself gain value as the home owner performs exterior and interior renovations that add to the price tag of the house. In maths we called this is **APPRECIATION.**

SIMPLE INTEREST

$$A = P(1 + in)$$

A = accumulated amount

P = original amount

n = number of periods

i = interest rate ($\frac{r}{100}$)

COMPOUND INTERES

$$A = P(1 + i)^n$$

Compounding periods	**n** (y = years)	**i**
Annually	y	i
Semi annually or Half yearly	$y \times 2$	$i \div 2$
Monthly	$y \times 12$	$i \div 12$
Quarterly	$y \times 4$	$i \div 4$

DEPRECIATION (DECAY)

STRAIGHT LINE METHOD

$$\boldsymbol{A = P(1 - in)}$$

A = book value/ scrap value

P = original amount

n = number of periods

i = interest rate ($\frac{r}{100}$)

REDUCING-BALANCE METHOD

$$\boldsymbol{A = P(1 - i)^n}$$

Compounding periods	n (y = years)	i
Annually	y	i
Semi annually or Half yearly	$y \times 2$	$i \div 2$
Monthly	$y \times 12$	$i \div 12$
Quarterly	$y \times 4$	$i \div 4$

NOMINAL VS EFFECTIVE INTEREST RATES

Annual effective rate is equivalent to the nominal rate per annum compounded monthly, because it produces the same accumulated amount.

$$\boxed{1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n}$$

i_{eff} = effective rate (annual)

i_{nom} = nominal rate

n = number of compoundings per year

GRADE 12

Future Value annuity

$$F = \frac{x[(1+i)^n - 1]}{i}$$

F = Future value
x = fixed regular payments
n = number of payments
i = interest rate in decimals

When there is "x" immediate payment made and the last payment is made at the end of the period:

$$F = \frac{x[(1+i)^{n+1} - 1]}{i}$$

When there is an immediate payment made of an amount that is not x, say t, and the last payment is made at the end of the period:

$$F = t(1+i)^n + \frac{x[(1+i)^n - 1]}{i}$$

When payments are made at the beginning of each period or when payments are made at the end of each period and the last payment is made, for an example 1 month before the end of the period if interest is compounded monthly:

$$F = \frac{x[(1+i)^n - 1]}{i} \times (1+i)^n$$

Sinking Fund

Sinking fund is an amount that is invested to replace something (e.g. Vehicle, machinery) in future. We use future value annuity to save money in regular intervals for the money to be used in future.

N.B ***Sinking fund = New price after inflation − Book value***

= APPRECIATION – DEPRECIATION

Present Value annuity

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$P =$ **Present value (loan amount)**
$x =$ **fixed regular payments**
$n =$ **number of payments**
$i =$ **interest rate in decimals**

Interest paid

Interest amount paid = ***All payments made − loan amount***

= Monthly payment×(n×compounding period) - loan amount

Balance on the loan	
Balance = $P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$ **OR** **Balance=** $\frac{x[(1+i)^{-m}+1]}{i}$	***where P is the loan amount.*** $Where\ m = n - number\ of\ paid\ periods$

PAPER A

QUESTION 7

7.1 Six years ago, Thabo bought a phone for R13 000. The value of the phone depreciated annually according to the reducing-balance method. The value of the phone is now R8 337,75. Calculate the annual rate of depreciation. (3)

7.2 Eric and Thandi need to save R80 000 each to go on a holiday at the end of December 2027.

- Thandi decides that she will start saving at the end of January 2025. She will make 36 monthly deposits into a savings account that pays interest at 8,6% p.a., compounded monthly. The deposit will be made at the end of each month.
- Eric calculates that if he makes 48 deposits of R1 402,31, starting at the end of January 2024, he will have enough money to go on holiday. He will make his deposits into a savings account at the end of each month. The savings account pays interest at 8,6% p.a., compounded monthly.

Calculate the difference between the total amount that Eric and Thandi will deposit into their respective savings accounts over the given period. (4)

7.3 Lesibana was granted a loan of R225 000. The rate of interest for the loan is 9% p.a., compounded monthly. Lesibana will make monthly payments of R5 500, starting exactly four months after the loan was granted. How many payments will Lesibana make to settle the loan? (6)

[13]

PAPER B

QUESTION 6

6.1 Patrick deposited an amount of R18 500 into an account earning r% interest p.a., compounded monthly. After 6 months, his balance was R19 319,48.

6.1.1 Calculate the value of r. (3)

6.1.2 Calculate the effective interest rate. (2)

6.2 Kuda bought a laptop for R10 000 on 31 January 2019. He will replace it with a new one in 5 years' time on 31 January 2024.

6.2.1 The value of the old laptop depreciates annually at a rate of 20% p.a. according to the straight-line method. After how many years will the laptop have a value of R0? (2)

6.2.2 Kuda will buy a laptop that costs R20 000. In order to cover the cost price, he made his first monthly deposit into a savings account on 28 February 2019. He will make his 60th monthly deposit on 31 January 2024. The savings account pays interest at 8,7% p.a., compounded monthly. Calculate Kuda's monthly deposit into this account. (4)

6.3 Tino wins a jackpot of R1 600 000. He invests all of his winnings in a fund that earns interest of 11,2% p.a., compounded monthly. He withdraws R20 000 from the fund at the end of each month. His first withdrawal is exactly 1 month after his initial investment. How many withdrawals of R20 000 will Tino be able to make from this fund? (5)

[16]

PAPER C

QUESTION 6

6.1 R12 000 was invested in a fund that paid interest at m% p.a., compounded quarterly. After 24 months, the value of the investment was R13 459.

Determine the value of m. (4)

6.2 On 31 January 2022, Tino deposited R1 000 in an account that paid interest at 7,5% p.a., compounded monthly. He continued depositing R1 000 on the last day of every month. He will make the last deposit on 31 December 2022.

Will Tino have sufficient funds in the account on 1 January 2023 to buy a computer that costs R13 000? Justify your answer by means of an appropriate calculation. (4)

6.3 Thabo plans to buy a car that costs R250 000. He will pay a deposit of 15% and take out a loan for the balance. The interest on the loan is 13% p.a., compounded monthly.

6.3.1 Calculate the value of the loan. (1)

6.3.2 The first repayment will be made 6 months after the loan has been granted. The loan will be repaid over a period of 6 years after it has been granted. Calculate the MONTHLY instalment. (5)

[14]

PAPER D

QUESTION 7

7.1 R1 570 is invested at 12% p.a. compound interest. After how many years will the investment be worth R23 000? (4)

7.2 A farmer has just bought a new tractor for R800 000. He has decided to replace the tractor in 5 years' time, when its trade-in value will be R200 000. The replacement cost of the tractor is expected to increase by 8% per annum.

7.2.1 The farmer wants to replace his present tractor with a new one in 5 years' time. The farmer wants to pay cash for the new tractor, after trading in his present tractor for R200 000. How much will he need to pay? (3)

7.2.2
- One month after purchasing his present tractor, the farmer deposited x rands into an account that pays interest at a rate of 12% p.a., compounded monthly.
- He continued to deposit the same amount at the end of each month for a total of 60 months.
- At the end of 60 months he has exactly the amount that is needed to purchase a new tractor, after he trades in his present tractor.

Calculate the value of x. (6)

7.2.3 Suppose that 12 months after the purchase of the present tractor and every 12 months thereafter, he withdraws R5 000 from his account, to pay for maintenance of the tractor. If he makes 5 such withdrawals, what will the new monthly deposit be? (4)

PAPER E

QUESTION 7

7.1 How many years will it take for an article to depreciate to half its value according to the reducing-balance method at 7% per annum? (4)

7.2 Two friends each receive an amount of R6 000 to invest for a period of 5 years. They invest the money as follows:

- Radesh: 8,5% per annum simple interest. At the end of the 5 years, Radesh will receive a bonus of exactly 5% of the principal amount.
- Thandi: 8% per annum compounded quarterly.

Who will have the bigger investment after 5 years? Justify your answer with appropriate calculations. (6)

7.3 Nicky opened a savings account with a single deposit of R1 000 on 1 April 2011. She then makes 18 monthly deposits of R700 at the end of every month. Her first payment is made on 30 April 2011 and her last payment on 30 September 2012. The account earns interest at 15% per annum compounded monthly.

Determine the amount that should be in her savings account immediately after her last deposit is made (that is on 30 September 2012). (6)

PAPER F

QUESTION 8

8.1 A new cellphone was purchased for R7 200. Determine the depreciation value after 3 years if the cellphone depreciates at 25% p.a. on the reducing-balance method. (3)

8.2 Jill negotiates a loan of R300 000 with a bank which has to be repaid by means of monthly payments of R5 000 and a final payment which is less than R5 000. The repayments start one month after the granting of the loan. Interest is fixed at 18% per annum, compounded monthly.

8.2.1 Determine the number of payments required to settle the loan. (6)

8.2.2 Calculate the balance outstanding after Jill has paid the last R5 000. (5)

8.2.3 Calculate the value of the final payment made by Jill to settle the loan. (2)

8.2.4 Calculate the total amount that Jill repaid to the bank. (1)

PAPER G

QUESTION 8

8.1 A car depreciated at the rate of 13,5 % p.a. to R250 000 over 5 years according to the reducing balance method. Determine the original price of the car, to the nearest rand. (3)

8.2 Melissa takes a loan of R950 000 to buy a house. The interest is 14,25 % p.a. compounded monthly. His first instalment will commence one month after taking out the loan.

8.2.1 Calculate the monthly repayments over a period of 20 years. (4)

8.2.2 Determine the balance on the loan after the 100^{th} instalment. (4)

8.2.3 If Melissa failed to pay the 101^{st}, 102^{nd}, 103^{rd} and 104^{th} instalments, calculate the value of the new instalment that will settle the loan in the same time period. (4)

PAPER H

QUESTION 8

8.1 An investor indicates that he will be able to treble the value of the investment at the end of 6 years. The interest rate is fixed and compounded monthly. Calculate the annual interest rate that the investor has on offer. (4)

8.2 Samuel decided to buy a car costing R192 000. He takes out a loan for 5 years at an interest rate charged at 12 % p.a. compounded monthly. Payments are made at the end of each month.

8.2.1 Calculate the monthly repayments over a period of 5 years. (4)

8.2.2 After Samuel had made 45 payments, he decides to settle the balance on the loan. Calculate the lump sum that he will need to pay off the loan after he has made the 45^{th} payment. (4)

PAPER I

QUESTION 4

4.1 How many years will it take for an investment of R3 000 to accumulate to R4 500, if it is invested at 8% p.a. compounded monthly? (4)

4.2 Bongani paid off a 20-year loan of R40 000. During the period of the loan the interest rate changed from 24% p.a. compounded monthly for the first five years to 18% p.a. compounded monthly for the remaining years.

4.2.1 Calculate the initial monthly payment before the interest rate changed. (4)

4.2.2 What is the outstanding balance of the loan after the FIRST five years? (4)

4.2.3 Determine the monthly payment after the interest rate changed. (4)

PAPER J

QUESTION 7

7.1 Sarah's investment earns interest at 11% p.a. compounded semi-annually.
Mary's investment earns an effective interest of 11,42% *p.a.*
Whose investment, Sarah's or Mary's, earns a higher rate of interest per annum. (3)

7.2 Buhle decided to start saving before retirement. She makes payments of R10 000 monthly into an account yielding 7,72% *p.a.* compounded monthly, starting on 1 November 2016 with a final payment on 1 April 2026.

7.2.1 Calculate how much will be in the savings account immediately after the last deposit is made. (4)

7.2.2 At the end of the investment period Buhle re-invested the full amount in order for her to be able to draw a monthly pension from the fund.

She re-invested the money at an interest rate of 10% p.a. compounded monthly. If she draws an amount of R30 000 per month from this investment, for how many full months will she be able to receive R30 000? (4)

7.2.3 After withdrawing R30 000 for 20 months Buhle requires R1 500 000. Determine whether she can access this amount of money from this annuity. (4)

DIFFERENTIAL CALCULUS

The central concept of differential calculus is the *derivative*, which is an outgrowth of the velocities and slopes of tangents.

There are two ways to find the derivatives , from first principle and also by the use of the rules.

The formula $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is used to find any of the following from **FIRST PRINCIPLES**

or by **USE OF THE DEFINITION :**

- The derivative of f at any point.
- The gradient of the tangent to graph f at any point.
- The instantaneous rate of change of a function at any point.
- The gradient of a function at any point.

You must be able to find the derivatives of the following types for exam purpose:

$f(x) = c,$ $\quad f(x) = ax^2 + bx + c,$ $\quad f(x) = ax^3,$ $\quad f(x) = \frac{a}{x}$

Rules of differentiation

In grade 12 we use one rule, the power rule /constant multiple rule:

$$\boxed{\frac{d}{dx}\left(ax^n\right) = anx^{n-1} \quad for\ n \in \mathrm{R}}$$

which means differentiating the function (ax^n) with respect to x.

> **NOTE:** The notation we use for the derivative of $y = f(x)$ is
>
> $f'(x)$ or y' or $\frac{dy}{dx}$ or $D_x[f(x)]$.
>
> When we find the derivative of a function, we say we **differentiate** the function.

<u>Finding the equation of a tangent line</u>

The slope of the tangent line to the graph at a point is equal to the derivative of the function at that point. So, to find the equation of the tangent line to $f(x)$ at $x = a$, we must:

1. Find the derivative $f'(x)$

2. Work out the derivative at $x = a \rightarrow$ i.e calculate $f'(a)$ to get the gradient of the tangent line.

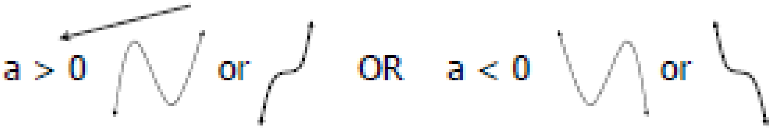
3. Calculate the y value at $x = a \rightarrow$i.e calculate $f(a)$.

4. The tangent line is a straight line. We can find the equation of a straight line using $y - y_1 = m(x - x_1)$

Steps to follow when sketching:

Standard form:

$y = ax^3 + bx^2 + cx + d \longrightarrow$ y-intercept



(N.B sometimes, if not most of the time, you will be directed by the question as to where to start,.. see the worked example)

1. Determine the x and $y - intercepts$
    a. x-intercept
        i. Let y = 0 and simplify
        ii. Find the factor
        iii. Solve for x
    b. y-intercept
        i. Let = 0 and solve for y x
2. Determine the turning points
    a. Find the first derivative
    b. Equate to zero
    c. Solve for x (these are x-values of the turning points)
    d. Substitute the x-values into the original function to find the corresponding y-values)
3. Sketch

PAPER A

QUESTION 8

8.1 Determine $f'(x)$ from first principles if $f(x)=\frac{1}{x}$. (5)

8.2 Determine:

8.2.1 $\frac{d}{dx}\left(\sqrt{4x^6}+\sqrt{2}.x^2\right)$ (3)

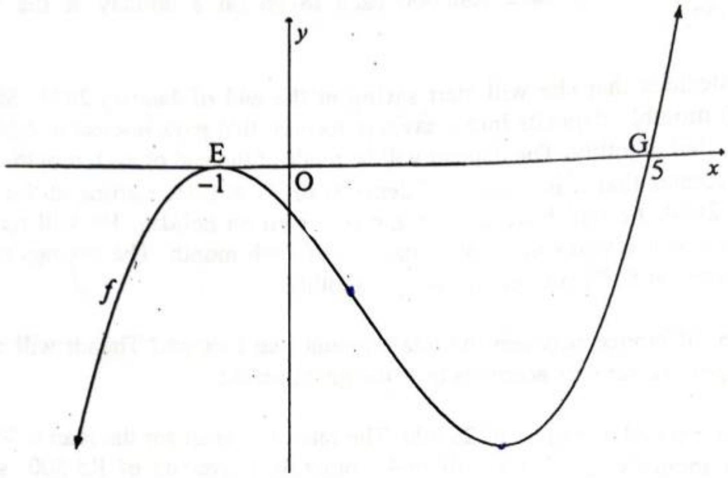
8.2.2 $g'(x)$ if $g(x)=\frac{3x^4-4x^2+6}{x^2}$ (3)

8.3 The equation of the tangent to $f(x)=3x^2+bx+c$ at $x=1$ is given by $y=9x-9$. Determine the values of b and c. (4)

[15]

QUESTION 9

The graph of $f(x)=ax^3+bx^2+cx-5$ is drawn below. E$(-1\ ;\ 0)$ and G$(5\ ;\ 0)$ are the x-intercepts of f.



9.1 Show that $a=1$, $b=-3$ and $c=-9$. (3)

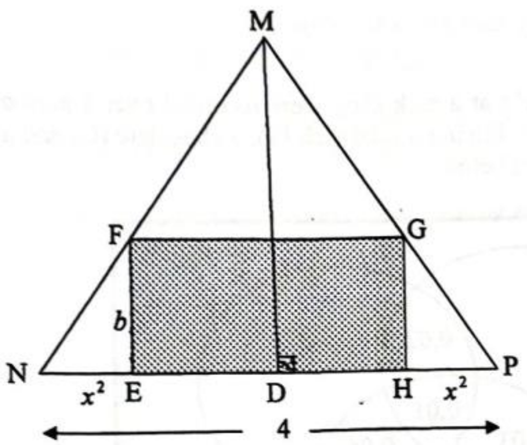
9.2 Calculate the value of x for which f has a local minimum value. (4)

9.3 Use the graph to determine the values of x for which $f''(x).f(x)>0$. (3)

9.4 For which values of t will the graph of $p(x)=f(x)+t$ have two distinct positive roots and one negative root? (3)

QUESTION 10

EHGF is a rectangle. HE is produced x^2 cm to N and EH is produced x^2 cm to P. NF produced intersects PG produced at M to form an isosceles triangle MNP with $NM = MP$. D lies on NP where $MD \perp NP$. $NP = 4$ cm and $MD = 3$ cm.



10.1 Show that the area of EFGH is given by $A(x) = 6x^2 - 3x^4$. (4)

10.2 Calculate the maximum area of rectangle EFGH. (4)
[8]

PAPER B

QUESTION 7

7.1 Determine $f'(x)$ from first principles if $f(x) = -4x^2$ (5)

7.2 Determine:

7.2.1 $f'(x)$ if $f(x) = 2x^3 - 3x$ (2)

7.2.2 $D_x\left(7.\sqrt[3]{x^2} + 2x^{-5}\right)$ (3)

7.3 For which values of x will the tangent to $f(x) = -2x^3 + 8x$ have a positive gradient? (3)
[13]

QUESTION 8

Given: $f(x)=-x^3+6x^2-9x+4=(x-1)^2(-x+4)$

8.1 Determine the coordinates of the turning points of f. (4)

8.2 Draw a sketch graph of f. Clearly label all the intercepts with the axes and any turning points. (4)

8.3 Use the graph to determine the value(s) of k for which $-x^3+6x^2-9x+4=k$ will have three real and unequal roots. (2)

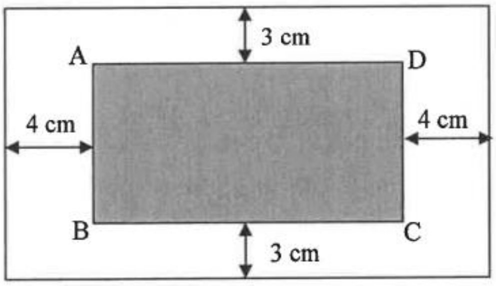
8.4 The line $g(x)=ax+b$ is the tangent to f at the point of inflection of f. Determine the equation of g. (6)

8.5 Calculate the value of θ, the acute angle formed between g and the x-axis in the first quadrant. (2)

[18]

QUESTION 9

The diagram below represents a printed poster. Rectangle ABCD is the part on which the text is printed. This shaded area ABCD is $432\ \text{cm}^2$ and $\text{AD}=x$ cm. ABCD is 4 cm from the left and right edges of the page and 3 cm from the top and bottom of the page.



9.1 Show that the total area of the page is given by:

$$A(x)=\frac{3\,456}{x}+6x+480$$

(3)

9.2 Determine the value of x such that the total area of the page is a minimum. (3)

[6]

PAPER C

QUESTION 9

9.1 Determine $f'(x)$ from first principles given $f(x) = x^2 - \frac{1}{2}x$. (5)

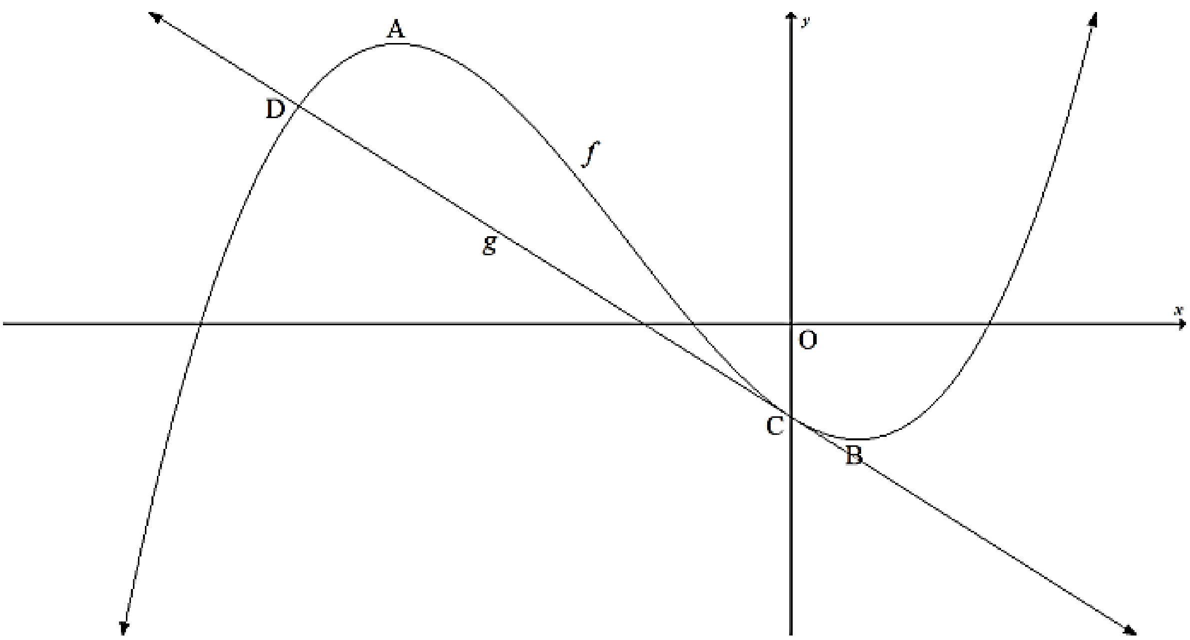
9.2 Determine:

9.2.1 $\frac{d}{dx}[3x^4 + \sqrt[5]{x} + a^2]$ (3)

9.2.2 $\frac{dy}{dx}$, if $xy = x + x^2 - 1$. (4)

QUESTION 10

In the diagram, the graph of $f(x) = x^3 + 5x^2 - 8x - 12$ is drawn. A and B are the turning points and C the y - intercept of f. $g(x) = mx + c$ is a tangent to the graph of f at C. D is the intersection of f and g.



10.1 Calculate the:

10.1.1 co-ordinates of the x-intercepts of f. (6)
10.1.2 co-ordinates of B. (4)
10.1.3 x – coordinate of the point of inflection of f. (2)

10.2 Determine the:

10.2.1 equation of the g. (2)
10.2.2 values of x for which $f'(x). g'(x) > 0$. (3)

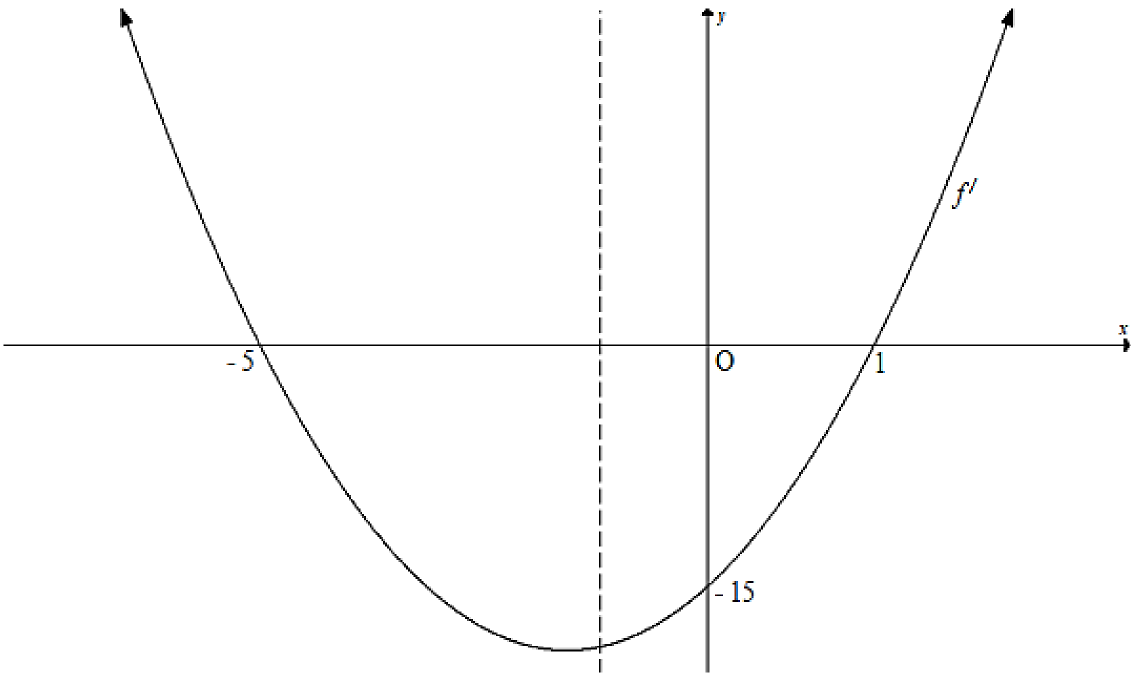
PAPER D

QUESTION 9

9.1	Determine $f^{/}(x)$ from first principles given $f(x) = x^2 - bx$, where b is a constant.	(5)
9.2	Determine:	
9.2.1	$\dfrac{d}{dx}\left[\dfrac{x^4}{4} - 3.\sqrt[3]{x} + 7\right]$	(3)
9.2.2	$\frac{dy}{dx}$ if $y = (x^{\frac{1}{3}} - 2x^{\frac{2}{3}})^2$	(4)

QUESTION 10

The graph of $f^{/}$, the derivative of f, is drawn below. $f(x) = ax^3 + bx^2 + cx + d\,;\, a \neq 0.$
$f^{/}$ intersects the $x-$axis at -5 and 1 and the $y-$axis at -15.

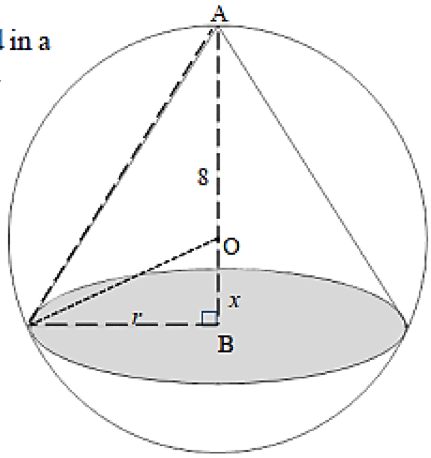


10.1	Write down the:	
	10.1.1 $x-$values of the turning points of f.	(2)
	10.1.2 $x-$value(s) where the gradient of f is equal to -15.	(2)
10.2	Show that the equation of $f^{/}$ is given by $y = 3x^2 + 12x - 15$.	(3)
10.3	If $f(-3) = 0$, calculate the value of d.	(4)
10.4	Determine the coordinates of the turning points of the graph of f and state whether they are maximum or minimum turning points.	(4)
10.5	$y = tx + 4$ is a tangent to f. Calculate the value of t.	(5)

PAPER E

QUESTION 9

A cone with radius r cm and height AB is inscribed in a sphere with centre O and a radius of 8 cm. $\text{OB} = x$.



Volume of sphere $= \frac{4}{3}\pi r^3$
Volume of cone $= \frac{1}{3}\pi r^2 h$

9.1 Calculate the volume of the sphere. (1)

9.2 Show that $r^2 = 64 - x^2$. (1)

9.3 Determine the ratio between the largest volume of this cone and the volume of the sphere. (7)

QUESTION 10

A cubic function has the following essential properties:

- $f(0) = 8$
- $f(4) = f(1) = 0$
- $f'(3) = f'(1) = 0$
- $f(3) = 8$

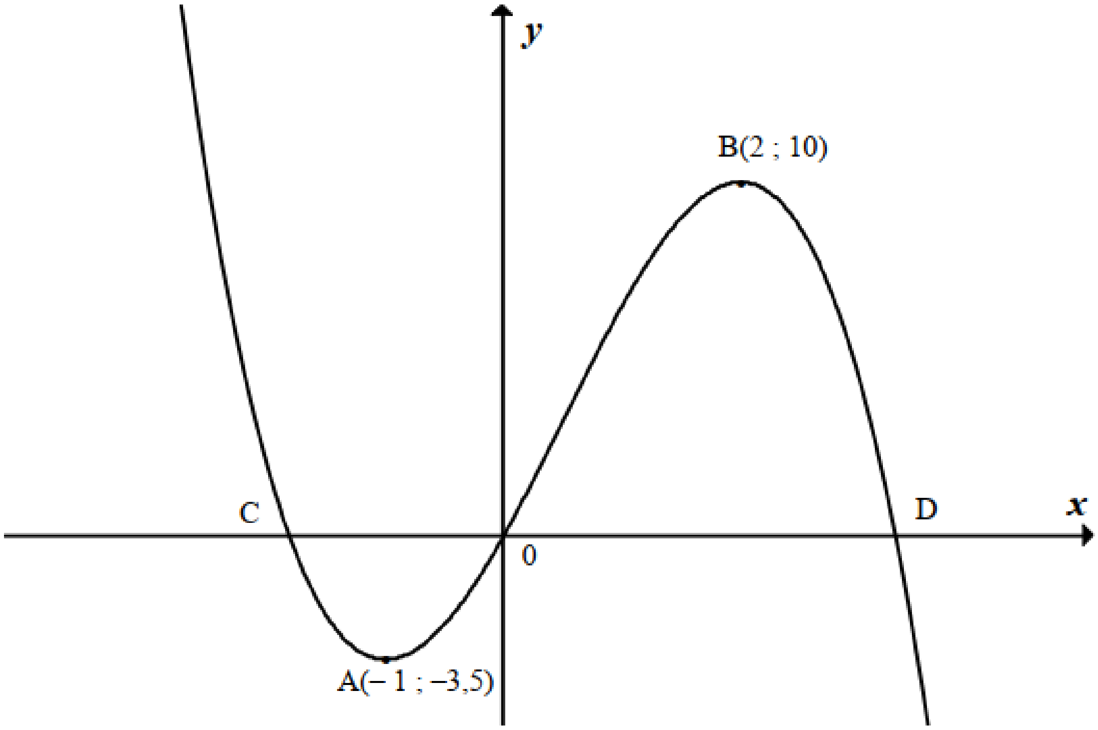
10.1 Sketch the graph of f in your ANSWER BOOK clearly indicating the turning point(s) and the points of intersection of the graph with the axes. (3)

10.2 Show that the defining equation of f is $f(x) = -2x^3 + 12x^2 - 18x + 8$. (4)

10.3 Determine the value(s) of x for which graph of f is concave down. (3)

QUESTION 12

The graph of $h(x) = -x^3 + ax^2 + bx$ is shown below. A(-1 ; 3,5) and B(2 ; 10) are the turning points of h. The graph passes through the origin and further cuts the x-axis at C and D.



12.1 Show that $a = \frac{3}{2}$ and $b = 6$. (6)

12.2 Calculate the average gradient between A and B. (2)

12.3 Determine the equation of the tangent to h at $x = -2$. (5)

12.4 Determine the x-value of the point of inflection of h. (3)

12.5 Use the graph to determine the values of p for which the equation $-x^3 + \frac{3}{2}x^2 + 6x + p = 0$ will have ONE real root. (2)

PAPER F

QUESTION 8

8.1 If $f(x) = 2x^2 - 5x + 3$, determine $f'(x)$ from first principles. (5)

8.2 Determine $\frac{dy}{dx}$ if $y = \frac{2x^2}{3\sqrt{x}} - \frac{2x^3 + 1}{x^3}$ (5)

QUESTION 9

9.1 Given: $f(x) = -2x^3 + 5x^2 + 4x - 3$

9.1.1 Calculate the coordinates of the x-intercepts of f if $f(3) = 0$. Show all calculations. (4)

9.1.2 Calculate the x-values of the stationary points of f. (4)

9.1.3 For which values of x is f concave up? (2)

9.2 The function g, defined by $g(x) = ax^3 + bx^2 + cx + d$ has the following properties:

- $g(-2) = g(4) = 0$
- The graph of $g^{/}(x)$ is concave up.
- The graph of $g^{/}(x)$ has x-intercepts at $x = 0$ and $x =$ 4 and a turning point at $x = 2$.

9.2.1 Use this information to draw a neat sketch graph of g without actually solving for a, b, c and d. Clearly show all x-intercepts, x-values of the turning points and x-value of the point of inflection on your sketch. (4)

9.2.2 For which values of x will $g(x) \cdot g^{//}(x) > 0$? (3)

QUESTION 10

A shopkeeper finds that the number of people visiting his shop at any moment during the 10 hours that the shop is open, is represented by:

$$N(t) = t^3 - 12t^2 + 36t + 8 \ ,$$

where $N(t)$ is the number of people in the shop, t hours after the shop opened.

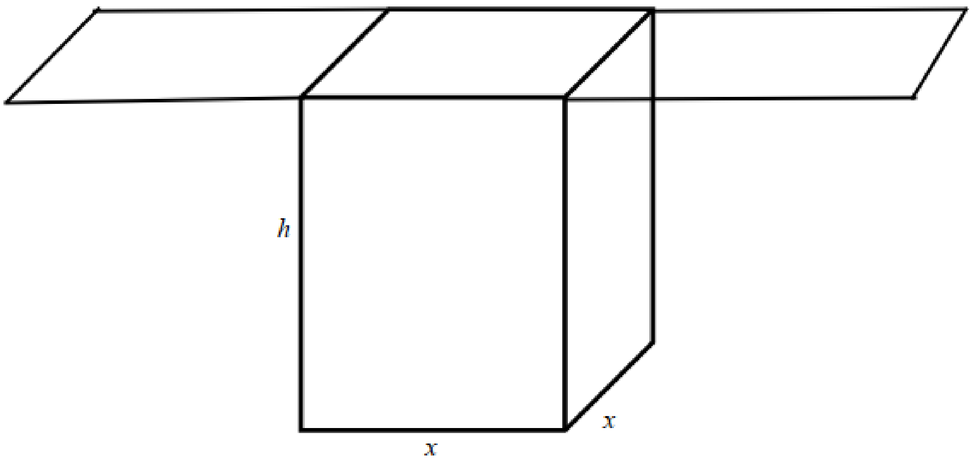
10.1 How many people are in the shop when the shop opens? (1)

10.2 At what stage is the number of people in the shop increasing? (5)

10.3 At which stage is it the best time for the shopkeeper to take a break and leave his assistant alone in the shop? (1)

QUESTION 11

The rectangular milk carton has a square base which holds 1 litre of milk. It has a specially designed fold-in top. The area of the cardboard used for the top is three times the area of the base.



11.1 Show that the Total Surface Area of the carton is given by (3)

$$A(x) = 4x^2 + \frac{4000}{x}$$

11.2 Determine the dimensions of the carton so that minimum amount of cardboard is used. (6)

PAPER G

QUESTION 8

8.1 Given $f(x) = 3 - 2x^2$. Determine $f'(x)$, using first principles. (5)

8.2 Determine $\frac{dy}{dx}$ if $y = \frac{12x^2 + 2x + 1}{6x}$. (4)

8.3 The function $f(x) = x^3 + bx^2 + cx - 4$ has a point of inflection at (2 ; 4). Calculate the values of b and c. (7)

QUESTION 9

Given: $f(x) = x^3 - x^2 - x + 1$

9.1 Write down the coordinates of the y-intercept of f. (1)

9.2 Calculate the coordinates of the x-intercepts of f. (5)

9.3 Calculate the coordinates of the turning points of f. (6)

9.4 Sketch the graph of f in your ANSWER BOOK. Clearly indicate all intercepts with the axes and the turning points. (3)

9.5 Write down the values of x for which $f'(x) < 0$. (2)

QUESTION 10

A particle moves along a straight line. The distance, s, (in metres) of the particle from a fixed point on the line at time t seconds ($t \geq 0$) is given by $s(t) = 2t^2 - 18t + 45$.

10.1 Calculate the particle's initial velocity. (Velocity is the rate of change of distance.) (3)

10.2 Determine the rate at which the velocity of the particle is changing at t seconds. (1)

10.3 After how many seconds will the particle be closest to the fixed point? (2)

PAPER H

QUESTION 8

8.1 If $f(x) = \frac{4}{x}$, determine $f'(x)$ from first principles. (5)

8.2 Determine:

8.2.1 $\frac{dy}{dx}$ if $y = 5x^2 + 5x + 2$ (2)

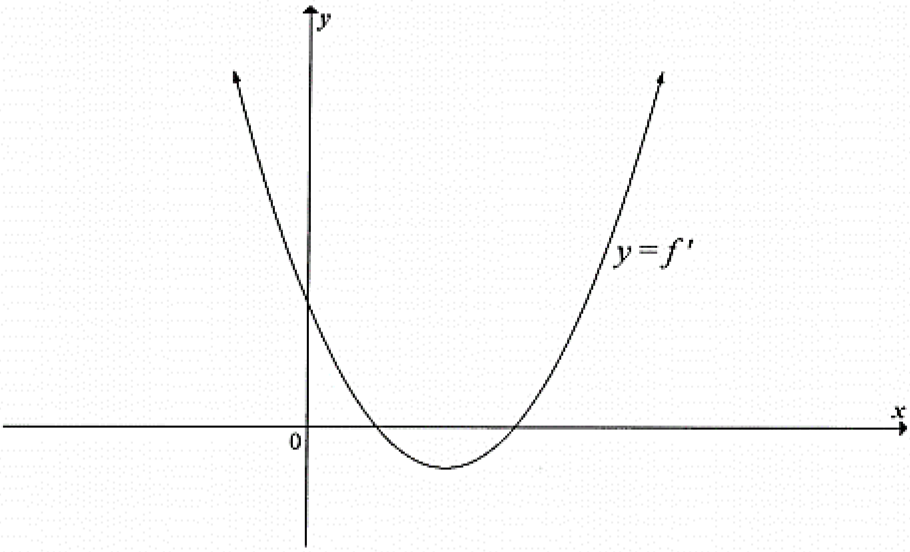
8.2.2 $D_x\left[\sqrt[3]{x^2} - \frac{1}{2}x\right]$ (3)

8.3 Given: $p(x) = x^3 + 2x$

Show, using relevant calculations, why it is not possible for a tangent drawn to the graph of p to have a negative gradient. (3)

QUESTION 9

The graph of $y = ax^2 + bx + c$ below represents the derivative of f.
It is given that $f'(1) = 0$, $f'(3) = 0$ and $f'(0) = 6$.



9.1	Write down the x-coordinates of the stationary points of f.	(2)
9.2	For which value(s) of x is f strictly decreasing?	(2)
9.3	Explain at which value of x the stationary point of f will be a local minimum.	(2)
9.4	Determine the x-coordinate of the point of inflection of f.	(1)
9.5	For which value(s) of x is f concave up?	(2)

QUESTION 10

The mass of a baby in the first 30 days of life is given by

$M(t) = t^3 - 9t^2 + 3\,000 \quad ; \quad 0 \leq t \leq 30$.

t is the time in days and M is the mass of the baby in grams.

10.1	Write down the mass of the baby at birth.	(1)
10.2	A baby's mass usually decreases in the first few days after birth. On which day will the baby's mass return to its birth mass?	(4)
10.3	On which day will this baby have a minimum mass?	(4)
10.4	On which day will the baby's mass be decreasing the fastest?	(2)

PROBABILITY AND COUNTING

<table>
<tr>
<td>
Theoretical Probability of an event happening :

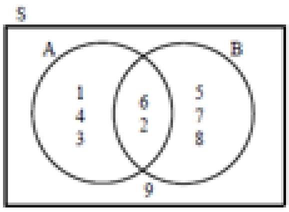
$$P(E) = \frac{\text{number of possible times event can occur}}{\text{number of possible outcomes}}$$

$$= \frac{n(E)}{n(S)}$$

E = Event and S = Sample space
</td>
<td>
Addition rule :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A and B are inclusive events as they have elements in common.



$A \cap B = \{2; 6\}$
$A \cup B = \{1; 2; 3; 4; 5; 6; 7; 8\}$
$n(A) = 5$
$n(B) = 5$
$n(A \cap B) = 2$
$n(A \cup B) = 8$
</td>
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<tr>
<td>
Mutually exclusive :

A and B are mutually exclusive events as they have no elements

in common.

$A \cap B = \{\}$

$P(A \cap B) = 0$
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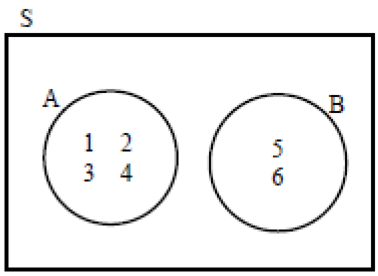
Exhaustive Events:

Events are exhaustive when they cover all elements in the sample

set.



</td>
</tr>
<tr>
<td>
Complimentary Events:

Events A and B are complimentary events if they are mutually

exclusive and exhaustive.

$$P(A) + P(A') = 1$$
</td>
<td>
Independent events

$$P(A \cap B) = P(A) \times P(B)$$
</td>
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</table>

Visual Tools used to calculate probability

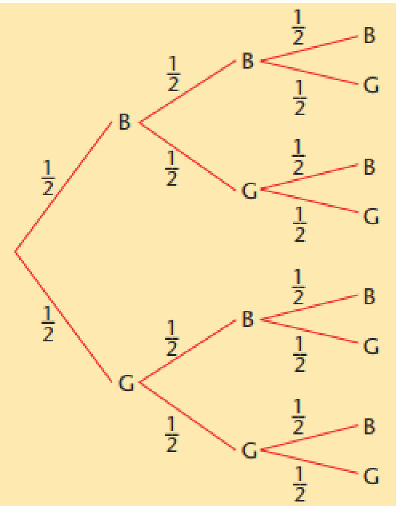
Tree diagrams

Useful when more than one events happen at the same time or one event happens after another event.

- Each event is represented by the column of branches.
- The number of branches is determined by the number of possible outcomes for the event.
- The probability must be shown in each branch.
- Sum of probabilities of each branch is 1.
- When determining a probability of particular outcome, read carefully whether the order of the happening of an event is given or not.

Tree diagram on independent events

Consider a family that plans to have three children. They can have a girl or a boy. A tree diagram will look like this:



Possible outcomes are BBB (all three children are boys), BBG, BGB, BGG, GBB, GBG GGB, GGG. There are three columns of branches since there are three events.

Calculating probabilities form a tree diagram

- To calculate probability of having three boys, P (BBB), we multiply probability of having a boy from first column of branches by the one of having a boy from the second column of branches by the one of having a boy from the third column of branches.

$$P\ (BBB) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Since the probabilities of the branches for each event are equal the same answer can be found using all possible outcomes. There is only one outcome of three boys out of 8 possible outcomes.

- Probability of having first and second born as boy means first and second born are boys, regardless of the gender of the third born. P(BBB) or P(BBG)

$P\ (BBB) +\ P\ (BBG) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Same answer can be found using outcomes. There are two possible ways a first and second born can be boys out of 8 possible outcomes.

- Probability of at least two girls means the outcome where there are two girls in any order or three girls. BGG or GBG or GGB or GGG

$$P\ (at\ least\ two\ girls) =\ P\ (BGG) +\ P\ (GGB) +\ P\ (GGB) +\ P\ (GGG) = \frac{4}{8}$$

$$= 1/2$$

- Probability of at most one girl means the outcome where there is one girl, in any order or no girl. BGG or GBG or GGB or GGG

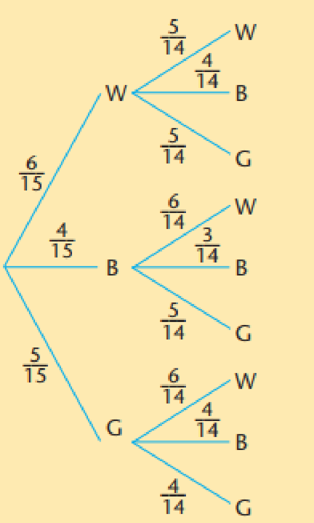
NB!

- ✓ **When sampling with replacement, such events are considered independent.**

Tree diagram on independent events

- For dependent events, the outcome of the successive event will be affected.
- The probabilities of the branches will not be the same. Therefore, probability cannot be calculated from the outcomes, it is calculated by multiplying the probabilities of the columns of branches.
- It applies on the problems where we sample/ choose an item without replacement.

Consider a drawer having 6 white cups, 4 black cups and 5 green cups. A cup is taken from a drawer and not replaced. Then the second cup is taken. Tree diagram will be as follows:



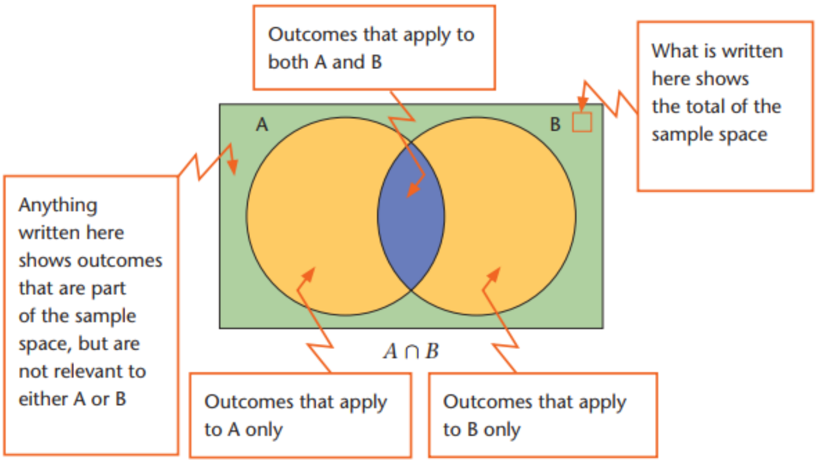
- Possible outcomes WW, WB, WG, BW, BB, BG, GW, GB, GG'
- If white cup is chosen first, it means for a second choice, there will be 5 cups remaining in the drawer and the total number of cups will be 14. The number of black and green cups will remain the same. That is why the top branch in second column has probability of 5/14, 4/14 and 5/14.
- Probability of both green cups;

P (GG) = $\frac{5}{15} \times \frac{4}{14} = \frac{2}{21}$.

Venn diagrams

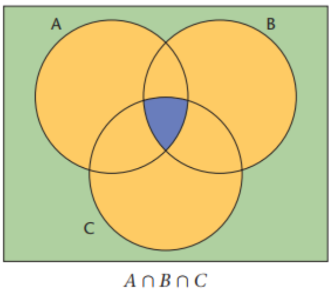
- Circle represents an event, if there are two circles that means there are two events.
- A rectangle represents the sample space.
- Read the statement correctly and look for key words like **A *only, A but not B, in both A and B, ALL.***
- Always start from central area (area about the intersection of all events), going out.
- After filling the numbers or probabilities on the circles, always add them. If the number is less than the sample size or probability is less than 1, then the difference is put outside the circle in the rectangle.

Venn diagram of two events.



- When drawing it, always start working with information on A and B.
- When given information on A, to fill the information about A only you subtract the number from A and B from the number of A.

Venn diagram of three events.



- Similar to Venn diagrams with two events, start working with an information from the central area (A and B and C) as shown on the diagram.
- Then move to areas about intersection of two events, A and B, B and C, A and C.
- Move to the regions of one event only.

Contingency table

Consider a 2-way contingency table below.

Gender	**Owning a cell phone**	**Not owning a cell phone**	**Total**
Male	a	b	**c**
Female	d	e	**f**
Total	**g**	**h**	**i**

- A letter a, represents the number of males owning a cell phone.
- A letter c, represents the total number of males.
- A letter g, represents total number of people owning cellphones.
- A letter i, represents the total number of participants in a survey.
- To determine the probabilities, we divide with i. E.g. Probability that a randomly selected person is male is c/i.
- The question may ask to test the independence of the events from contingency tables.

Counting principles

- Read the question properly to determine number of spaces needed.
- Check if repetition is allowed/ not allowed.
- Carefully, read if there is a restriction in groups (**… together**) or there is a restriction on the positions (particular seat must be occupied by a certain person).
- Always start from the positions/ groups with restrictions.
- Factorial works where we multiply numbers that are reduced by 1 successively, until we get to 1. E.g. Possible word arrangements from the word ACTIVE with no repetition of letters, seating 7 learners in a row, arranging 5 books on a table.
- When working with repeating letters, check a condition on repeating letters (identical or different). E.g. consider a word OMO where repeating letters are identical. Even though we have 2 O's, when choosing O, we only have **ONE** way of choosing it.
- Do not forget to remove repetition that will be caused by repeating letters by dividing with a product of factorials of repeating letters.
- When given a restriction on problems with repeating letters, start with restrictions, then re-write a new word with the remaining letters. Then find possible word arrangements.
- When calculating probabilities from words with repeating letters, you may assume the letters are different. BUT this only works when calculating the probabilities.
- To get a sample space, calculate the possible outcomes without any restriction given.

Given different choices c, d and e

$$n(s) = c \times d \times e$$

EXAMPLE:

How many different outfits could you put together with 4 shirts, 6 skits and 2 pairs of shoes?

$n(s) = 4 \times 6 \times 2$

$= 48$ outfits

Arrangements with repetition:

$$n(s) = k^x$$

Where;
k = number of choices
x = number of times you can choose

EXAMPLE:

How many ways van the letters in 'ERIN' be arranged with repetition?

$n(s) = 4^4$

$= 256$

EXAMPLE:

How many three letter codes can be made from the letters d, g, h, m, r, and t, if the letters can be repeated?

$n(s) = 6^3$

$= 216$

Arrangements without repetition:

$n(s) = p!$ (factorial notation)

$= p \times (p-1) \times (p-2) \times (p-3) \times \ldots$

EXAMPLE:

How many ways can the letters in Erin be arranged without repetition?

$n(s) = 4!$

$= 4 \times 3 \times 2 \times 1$

$= 24$

Order of arrangement is important (Permutation):

$$n(s) = \frac{n!}{(n-r)!} \quad or \quad n(s) = nPr$$

where;
r = number of specific choices

EXAMPLE:

There are 7 players in a netball team who hope to be shooter or goal attack. How many different options are there?

$n(s) = \dfrac{7!}{(7-2)!}$

$= 42$

or

$n(s) = 7P2$

$= 42$

Use $[nPr]$ key on calculator:

[7][nPr][2] [=]

Identical items (repetition) in an arrangement:

$$n(s) = \frac{n!}{m! \times p!}$$

where;
m and p: number of times different items are repeated

EXAMPLE:

How many times can the letters in the name 'VANESSA' be arranged?

There are 2 A's and 2 S's:

$n(s) = \dfrac{7!}{2! \times 2!}$

$= 1260$

Arrangements and Set Positions:

$n(s)$ = number of positions $\times$ number of arrangements in each position

EXAMPLE:

How many ways can 5 Maths books, 2 Afrikaans books and 3 English books be arranged if they are grouped in their subjects?

Number of positions = 3
Number of arrangements for Maths books = 5!
Number of arrangements for Afrikaans books = 2!
Number of arrangements for English books = 3!

$n(s) = 3! \times 5! \times 2! \times 3!$

$= 4\ 320$

EXAMPLE:

Questions:

A four-digit code can be made from four numbers 1 to 9 and 4 vowels.

1. How many possible codes can be made with repetition?
2. How many codes can be formed if the vowels cannot be repeated?
3. What is the probability of a code been created with no numbers been repeated?

Solutions:

1. $n(S) = n(\text{no. codes}) + n(\text{vowel codes})$
   $= 9^4 + 5^4$
   $= 7\ 186$
2. $n(E) = n(\text{no. codes}) + n(\text{vowel codes})$
   $= 9^4 + 5P4$
   $= 6\ 681$
3. $n(E_2) = n(\text{no. codes}) + n(\text{vowel codes})$
   $= 9P4 + 5^4$
   $= 3\ 649$

   $\therefore P(E_2) = \dfrac{n(E_2)}{n(S)}$

   $= \dfrac{3649}{7186}$

   $= 0{,}51$

EXAMPLE:

Questions:

A password consists of 8 characters. The first two characters must be any consonant and may not be repeated. The third letter is a vowel. The next four characters form a four-digit number which must not start with 0 but digits can repeat. The last character is a vowel which must be different from the first vowel.
For example: **HG E 2558 A**

1. How many different passwords are possible?
2. What is the Probability the code will have an even number between the letters and end with an A?

Solutions:

1. 26 letters - 5 vowels = 21 consonants
   0;1;2;...9 is 10 numbers

   $n(S) = 21 \times 20 \times 9 \times 10^3 \times 4$
   $= 75\ 600\ 000$
2. $n(E) = 21 \times 20 \times 4 \times 9 \times 10^2 \times 5 \times 1$
   $= 7\ 560\ 000$

   $\therefore P(E) = \dfrac{n(S)}{n(E)}$

   $= \dfrac{7\ 560\ 000}{75\ 600\ 000}$

   $= 0{,}1$

(Source : Mathsclinic grade 12)

BASIC PROBABILITY QUESTIONS AND/OR VENN DIAGRAMS

PAPER A

QUESTION 9

9.1 For any two events A and B, it is given that $P(\text{A}) = 0,35$ and $P(\text{A or B}) = 0,61$. Determine $P(\text{B})$ if:

9.1.1 A and B are mutually exclusive. (3)

9.1.2 A and B are independent. (4)

PAPER B

QUESTION 4

P(A) = 0,3 and P(B) = 0,5.

Calculate P(A or B) if:

4.1 A and B are mutually exclusive events (2)

4.2 A and B are independent events (3)

PAPER C

QUESTION 4

The events A, B and C are such: A and B are independent, B and C are independent and A and C are mutually exclusive. Their probabilities are P(A) = 0,3 , P(B) = 0,4 and P(C) = 0,2.

Calculate the probability of the following events occurring:

4.1 Both A and C occur. (2)

4.2 Both B and C occur. (2)

4.3 At least one of A or B occur. (4)

PAPER D

QUESTION 12

12.1 It is given that A and B are independent events. P(A) = 0,4 and P(B) = 0,5.

Use a Venn diagram and calculate:

12.1.1 P(A or B) (4)

12.1.2 P(neither A or B) (1)

PAPER E

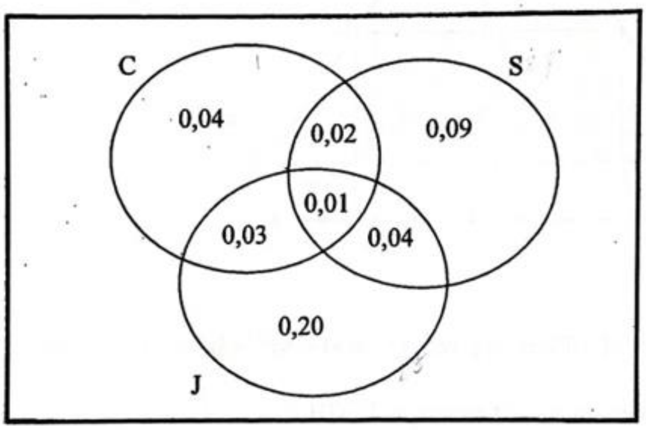
QUESTION 11

11.1 Two events, A and B, are such that:

- P(A) = 0,4
- P(A or B) = 0,52
- A and B are mutually exclusive

Calculate P(B). (2)

11.2 The items that a learner bought at a tuck shop were recorded over a period of time. The probabilities of the learner buying a sandwich (S), a chocolate (C) and a juice (J) are shown in the Venn diagram below.



11.2.1 What is the probability that the learner will buy a sandwich? (1)

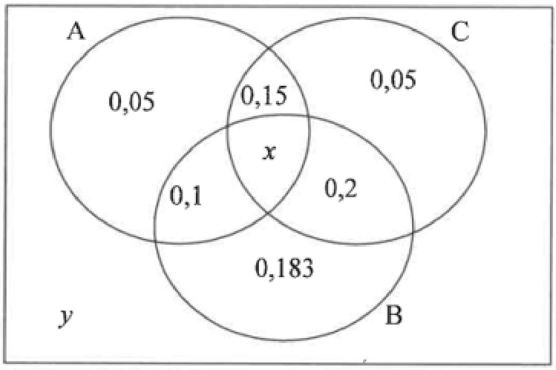
11.2.2 Calculate the probability that the learner will buy at least two of the three items. (2)

11.2.3 Calculate the probability that the learner would NOT buy any of the three items. (2)

PAPER F

QUESTION 10

10.1 A, B and C are three events. The probabilities of these events (or any combination of them) occurring is given in the Venn-diagram below



10.1.1 If it is given that the probability that at least one of the events will occur is 0,893, calculate the value of:

(a) y, the probability that none of the events will occur. (1)

(b) x, the probability that all three events will occur. (1)

10.1.2 Determine the probability that at least two of the events will take place. (2)

10.1.3 Are events B and C independent? Justify your answer. (5)

PAPER G

QUESTION 4

4.1 A survey of 80 students at a local library indicated the reading preferences below:

44 read the *National Geographic* magazine

33 read the *Getaway* magazine

39 read the *Leadership* magazine

23 read both *National Geographic* and *Leadership* magazines

19 read both *Getaway* and *Leadership* magazines

9 read all three magazines

69 read at least one magazine

4.1.1 How many students did not read any magazine? (1)

4.1.2 Let the number of students who read *National Geographic* and *Getaway*, but not *Leadership*, be represented by x. Draw a Venn diagram to represent reading preferences. (5)

4.1.3 Hence show that $x = 5$. (3)

4.1.4 What is the probability, correct to THREE decimal places, that a student selected at random will read at least two of the three magazines? (3)

4.2 A smoke detector system in a large warehouse uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0,95. The probability that it will be detected by device B is 0,98 and the probability that it will be detected by both devices simultaneously is 0,94.

4.2.1 If smoke is present, what is the probability that it will be detected by device A or device B or both devices? (3)

4.2.2 What is the probability that the smoke will not be detected? (1)

TREE DIAGRAMS

PAPER H

QUESTION 10

A bag contains x balls of which 5 are red and the rest are green. One ball is taken out of the bag randomly and it is not replaced. A second ball is taken out of the bag. The probability of picking both green balls is $\frac{3}{11}$. Show that the probability of picking both green balls can be represented by the equation: $4x^2 - 59x + 165 = 0$ **[4]**

PAPER I

QUESTION 10

10.1 A and B are independent events. $P(A) = \frac{1}{3}$ and $P(B) = \frac{3}{4}$

Determine:

10.1.1 P(A and B) (2)

10.1.2 P(at least ONE event occurs) (2)

10.2 The probability that it will snow on the Drakensberg Mountains in June is 5%.

- When it snows on the mountains, the probability that the minimum temperature in Central South Africa will drop below 0 °C is 72%.
- If it does not snow on the mountains, the probability that the minimum temperature in Central South Africa will drop below 0 °C is 35%.

10.2.1 Represent the given information on a tree diagram. Clearly indicate the probabilities associated with EACH branch. (3)

10.2.2 Calculate the probability that the temperature in Central South Africa will NOT drop below 0 °C in June 2024. (3)

PAPER J

QUESTION 3

The probability that it will rain on a given day is 63%. A child has a 12% chance of falling in dry weather and is three times as likely to fall in wet weather.

3.1 Draw a tree diagram to represent all outcomes of the above information. (6)

3.2 What is the probability that a child will not fall on any given day? (3)

3.3 What is the probability that a child will fall in dry weather? (2)

PAPER K

4.2 There are 20 boys and 15 girls in a class. The teacher chooses individual learners at random to deliver a speech.

4.2.1 Calculate the probability that the first learner chosen is a boy. (1)

4.2.2 Draw a tree diagram to represent the situation if the teacher chooses three learners, one after the other. Indicate on your diagram ALL possible outcomes. (4)

4.2.3 Calculate the probability that a boy, then a girl and then another boy is chosen in that order. (3)

4.2.4 Calculate the probability that all three learners chosen are girls. (2)

4.2.5 Calculate the probability that at least one of the learners chosen is a boy. (3)

PAPER L

QUESTION 5

Alfred and Barry have an equal chance of winning a point in a game.

5.1 Draw a tree diagram to represent the situation after a total of 3 points have been contested. Indicate on your diagram the probabilities and all the outcomes associated with each branch. (5)

5.2 Calculate the probability that Barry would have won all 3 points. (2)

5.3 Calculate the probability that Alfred would have won 2 points and Barry would have won 1 point of the 3 points contested. (2)

5.4 Barry and Alfred play a fourth point. Calculate the probability that Alfred will win 3 of the 4 points contested. (4)

CONTINGENCY TABLES

PAPER M

9.2 A cell phone distribution company investigated the number of defective phones that they obtain from two suppliers, Axis Phones and Direct Phones. They recorded their findings in a contingency table.

	Axis Phones	Direct Phones	Total
Defective	58	a	b
Not Defective	326	188	514
Total	384	c	600

9.2.1 Determine the values of a, b and c. (3)

9.2.2 Calculate the probability that a cell phone chosen at random is supplied by Direct phones. (1)

9.2.3 Calculate the probability that a cell phone chosen at random is Not Defective **OR** it is from Axis Phones and Defective. (3)

PAPER N

QUESTION 5

The sports director at a school analysed data to determine how many learners play sport and what the gender of each learner is. The data is presented in the table below.

	DO NOT PLAY SPORT	PLAY SPORT	TOTAL
Male	51	69	120
Female	49	67	116
Total	100	136	236

5.1 Determine the probability that a learner, selected at random, is:

5.1.1 Male (2)

5.1.2 Female and plays sport (2)

5.2 Are the events 'male' and 'do not play sport' mutually exclusive? Use the values in the table to justify your answer. (2)

5.3 Are the events 'male' and 'do not play sport' independent? Show ALL calculations to support your answer. (4)

PAPER O

QUESTION 5

In a survey 1 530 skydivers were asked if they had broken a limb. The results of the survey were as follows:

	Broken a limb	Not broken a limb	TOTAL
Male	463	b	782
Female	a	c	d
TOTAL	913	617	1 530

5.1 Calculate the values of a, b, c and d. (4)

5.2 Calculate the probability of choosing at random in the survey, a female skydiver who has not broken a limb. (2)

5.3 Is being a female skydiver and having broken a limb independent? Use calculations, correct to TWO decimal places, to motivate your answer. (4)

PROBABILITY AND COUNTING

PAPER P

11.3 Seven guitar players, each with a different name, participate in a concert.

11.3.1 In how many different ways can the names of the guitar players be listed, one below the other, in the programme? (1)

11.3.2 After the performance, the guitar players wait backstage. There is a bench with only room for four to sit on.

What will be the probability that the four guitar players will be sitting in alphabetical order, from left to right? (3)

11.3.3 During the performance, the seven guitar players sit in a line on stage. Four guitar players are female and three are male.

In how many different ways can they be seated if the males may not sit next to each other? (3)

PAPER Q

10.3 Ten learners stand randomly in a line, one behind the other.

10.3.1 In how many different ways can the ten learners stand in the line? (1)

10.3.2 Calculate the probability that there will be 5 learners between the 2 youngest learners in the line. (4)

PAPER R

10.2 A four-digit code is required to open a combination lock. The code must be even-numbered and may not contain the digits 0 or 1. Digits may not be repeated.

10.2.1 How many possible 4-digit combinations are there to open the lock? (3)

10.2.2 Calculate the probability that you will open the lock at the first attempt if it is given that the code is greater than 5 000 and the third digit is 2. (5)

PAPER S

10.2 A FIVE-digit code is created from the digits 2 ; 3 ; 5 ; 7 ; 9.

How many different codes can be created if:

10.2.1 Repetition of digits is NOT allowed in the code (2)

10.2.2 Repetition of digits IS allowed in the code (1)

PAPER T

QUESTION 7

Consider the digits 1, 2, 3, 4, 5, 6, 7 and 8 and answer the following questions:

7.1 How many 2-digit numbers can be formed if repetition is allowed? (2)

7.2 How many 4-digit numbers can be formed if repetition is NOT allowed? (3)

7.3 How many numbers between 4 000 and 5 000 can be formed? (3)

PAPER U

QUESTION 12

12.1 A password consists of five different letters of the English alphabet. Each letter may be used only once. How many passwords can be formed if:

12.1.1 All the letters of the alphabet can be used (2)

12.1.2 The password must start with a 'D' and end with an 'L' (2)

PAPER V

QUESTION 5

Every client of CASHSAVE Bank has a personal identity number (PIN) which is made up of 5 digits chosen from the digits 0 to 9.

5.1 How many personal identity numbers (PINs) can be made if:

5.1.1 Digits can be repeated (2)

5.1.2 Digits cannot be repeated (2)

5.2 Suppose that a PIN can be made up by selecting digits at random and that the digits can be repeated. What is the probability that such a PIN will contain at least one 9? (4)

PAPER W

5.2 A photographer has placed six chairs in the front row of a studio. Three boys and three girls are to be seated in these chairs.

In how many different ways can they be seated if:

5.2.1 Any learner may be seated in any chair (2)

5.2.2 Two particular learners wish to be seated next to each other (3)

PAPER X

QUESTION 6

There are 7 different shirts and 4 different pairs of trousers in a cupboard. The clothes have to be hung on the rail.

6.1 In how many different ways can the clothes be arranged on the rail? (2)

6.2 In how many different ways can the clothes be arranged if all the shirts are to be hung next to each another and the pairs of trousers are to be hung next to each another on the rail? (3)

6.3 What is the probability that a pair of trousers will hang at the beginning of the rail and a shirt will hang at the end of the rail? (4)

PAPER Y

QUESTION 5

Consider the word: PRODUCT.

5.1 How many different arrangements are possible if all the letters are used? (2)

5.2 How many different arrangements can be made if the first letter is T and the fifth letter is C? (2)

5.3 How many different arrangements can be made if the letters R, O and D must follow each other, in any order? (3)

PAPER Z

QUESTION 11

11.1 The letters of the word EQUATION are randomly used to form a new word consisting of five letters. How many of these words are possible if letters may not be repeated? (2)

Question 6

Consider the letters of the word MAREMATLOU

6.1 What is the probability that the word arrangement formed with the letters, MAREMATLOU, will start and end with the same letter? Repeated letters are treated as identical. (5)

PAPER 2

STATISTICS

A. Candidates should be encouraged to use the calculator to calculate standard deviation, variance and the equation of the least squares regression line.

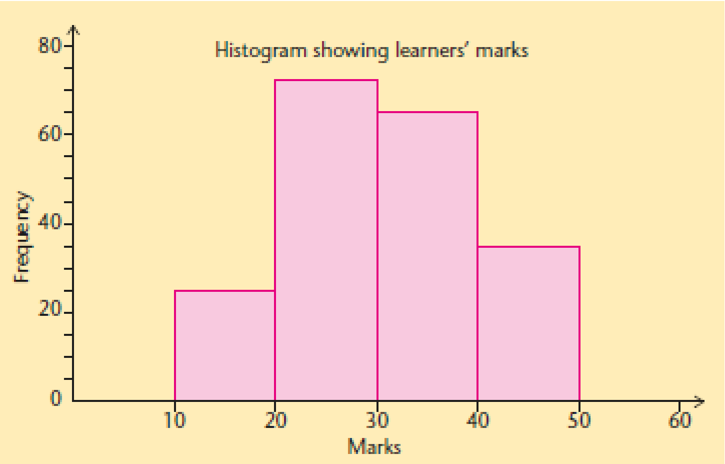
B. The interpretation of standard deviation in terms of normal distribution is not examinable.

1. DRAW **HISTOGRAMS**

A histogram gives us a visual interpretation of **GROUPED DATA**. It looks very similar to a **bar graph**, but there are **NO** gabs between the bars.

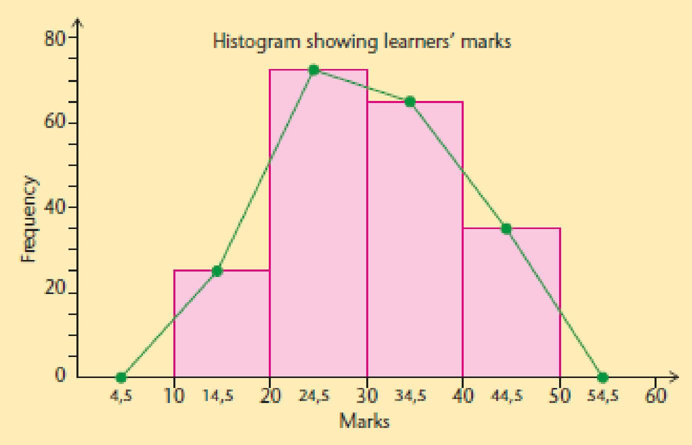
In HISTOGRAM INCLUDE THE FOLLOWING

- Tittle on top that describe what is contained in histogram
- Group/class intervals in x axis
- Frequency in y axis
- Bars with no gabs in between



2. DRAW **FREQUENCY POLYGONS**

➢ Drawn from **HISTOGRAM** by joining the **midpoints** of the top of the columns of the histogram. At the ends, extend line to the midpoints of class below lower values and the midpoint of the class above upper value to touch x axis**(grounded)**



- It can also be **drawn** by calculating midpoints of interval and plot coordinate **(midpoint, frequency)**

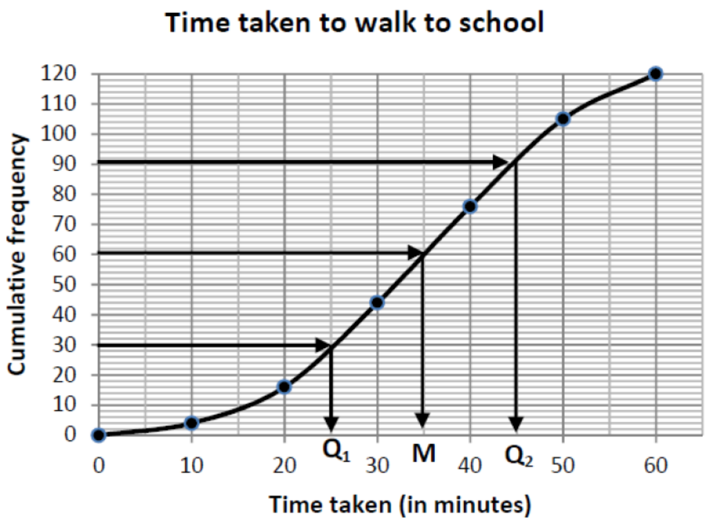
3. DRAW **OGIVES** (CUMULATIVE FREQUENCY CURVES)

Cumulative frequency shows the number of results that are less.

- To find the **cumulative frequency,**
  - Add up the frequencies as you go down the frequency table.
  - And the last **cumulative frequency** is equal to the total frequency? (This is a useful check of your addition.)

An **ogive** or **cumulative frequency curve** is a graph that shows the information in a cumulative frequency table.

Always remember when **drawing cumulative frequency** curve from a table of **grouped data**, the cumulative frequencies are plotted at the **upper limit** of the interval. **Grounded is important.**



4. **MEASURES OF CENTRAL TENDENCY FOR UNGROUPED DATA**;

$Mean = \dfrac{sum\ of\ all\ values}{total\ number\ of\ values}$ $\overline{x} = \dfrac{\sum x}{n}$	Where : $\overline{x}$ = mean Σx = sum of all values n = number of values

Mode

The mode is the value that appears most frequently in a set of data points.

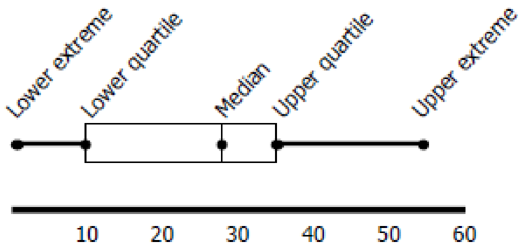
Median

The median is the middle number in a set of data points. position of median = $\frac{1}{2}(n + 1)$

Where; n = number of values

If n = odd number, the median is part of the data set.

If n = even number, the median will be the average between the two middle numbers.

FIVE NUMBER SUMMARY	**BOX AND WHISKER PLOT**
1. Minimum value 2. Lower quartile Q_1 3. Median 4. Upper quartile Q_3 5. Maximum value	A box and whisker plot is a visual representation of the five number summary. 

5. DETERMINE WHETHER DATA IS **SYMMETRIC** OR **SKEWED.**

- A **measure of shape** describes the distribution of the data within a data set.
- **Frequency polygons** and **box-and-whisker diagrams** can be used to illustrate symmetric and skewed data.

➢ KEY FEATURES OF A SYMMETRIC DISTRIBUTION

- The shape is symmetrical
- **The mode, median and mode have the same value.**
- **Most of the data are clustered around the centre.**

➢ KEY FEATURES OF SKEWED DATA

Skewness is the tendency for the values to be more frequently around the high or low ends of the x-axis.

➢ Note that if the mean and the median of a data set are known, then

- If ***mean – median* = *0,*** then the distribution is ***symmetric***
- If ***mean – median* > *0,*** then the distribution is ***positively skewed***
- If ***mean – median* < *0,*** then the distribution is ***negatively skewed***

➢ **Note that for a box-and-whisker diagram**

- If the distribution is **symmetric**, the median is in the middle of the box and the whiskers are equal in length
- When data is more spread out on the left side and clustered on the right, the distribution is said to be **negatively skewed** or **skewed to the left.**
- When the data is more spread out on the right side clustered on the left, the distribution is said to be positively skewed or skewed to the right.

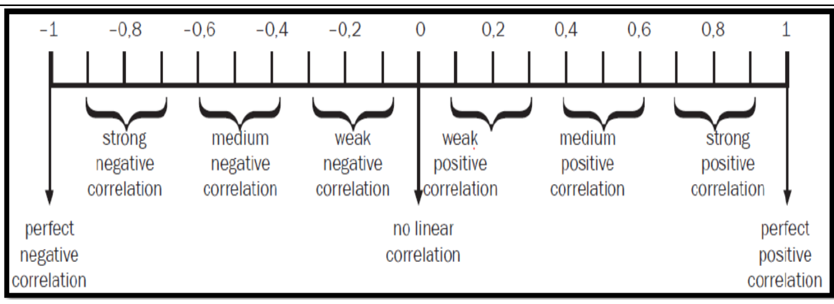
6. IDENTIFY THE VALUES OF THE **OUTLIERS**.

- An ***outlier*** is a data entry that is ***far removed from the other entries*** in the data set e.g. a data entry that is much smaller or much larger than the rest of the data values.
- An outlier has an influence on the ***mean*** and the ***range*** of the data set, but has no influence on the median or lower or upper quartiles.
- An outlier can affect the ***skewness*** of the data.
- Any data item that is

Less than $Q_1 - 1,5 \times$ IQR OR **More than $Q_3 + 1,5 \times$ IQR** is an outlier.

7. SCATTER PLOT ANALYSIS

Correlation coefficient

Where r indicates the strength of the relationship between the two variables (x and y). Properties: • The correlation coefficient is a number between –1 and 1 ($-1 \leq r \leq 1$)	

REGRESSION LINE (line of best fit)

If there is a strong linear correlation, a straight line may be drawn that act as a prediction of the values by looking at the trend. That line called line of best fit or the regression line or the least squares regression line. This line exclude outliers- values that are far away from line (Only if it is stated from the question). Your line of best-fit may not be the same as someone else's. Line of best fit must passes through mean point and the *y* intercept.

The equation of regression line is given by $\boldsymbol{\hat{y} = a + bx}$.

LINE OF BEST FIT CAN BE USED TO ESTIMATE.

✓ **INTERPOLATION –** Estimate the number **inside** the given period **e.g 10,5 ≈ 77 books**

✓ **EXTRAPOLATION -** Estimate the number **outside** the given period **e.g 13 ≈ 94 books**

HINT: Calculator may also be used to get mean point $(\overline{x}, \overline{y})$,.

8. CALCULATING THE MEAN AND STANDARD DEVIATION FOR GROUPED DATA.

Continuous data is grouped into class intervals which consist of an upper class boundary (maximum value) and lower class values (minimum value).

Class interval	frequency (f)	Midpoint $x = \frac{\text{uper class barrier} + \text{lower class barrier}}{2}$	$(f \times x)$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
$0 \le x \le 10$	3	$\frac{10+0}{2} = 5$	$3 \times 5 = 15$	$(5 - \overline{15{,}71})^2 = 114{,}7$	$3(114{,}7) = 344{,}11$
$10 \le x \le 20$	7	$\frac{20+10}{15} = 15$	$7 \times 15 = 105$	$(15 - \overline{15{,}71})^2 = 0{,}5$	$7(0{,}5) = 3{,}53$
$20 \le x \le 30$	4	$\frac{30+20}{2} = 25$	$4 \times 25 = 100$	$(25 - \overline{15{,}71})^2 = 88{,}3$	$4(88{,}3) = 354{,}22$
total :	**14**	**14**	**220**		$\boldsymbol{\Sigma f(x - \bar{x})^2 = 692{,}86}$

Mean:

$$\text{mean}(\bar{x}) = \frac{\text{sum of all frequency} \times \text{mean value}}{\text{total frequency}}$$

$$= \frac{220}{14}$$

$$= 15{,}71$$

Standard deviation:

$$\sigma = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{692{,}86}{14}}$$

$$= 7{,}03$$

A small Standard deviation tells us the numbers are clustered closely around the mean, a larger standard deviation indicates more scattered data.

CALCULATOR STEPS:

Mode 2: STAT 1: VAR

Shift: Setup screendown. 3: STAT

Frequency? 1: ON 2: OFF

Grouped data

Enter data (x = mid pt)

AC

Shift 4: VAR

2: $\bar{x}$ (mean)

3: σx (std deviation)

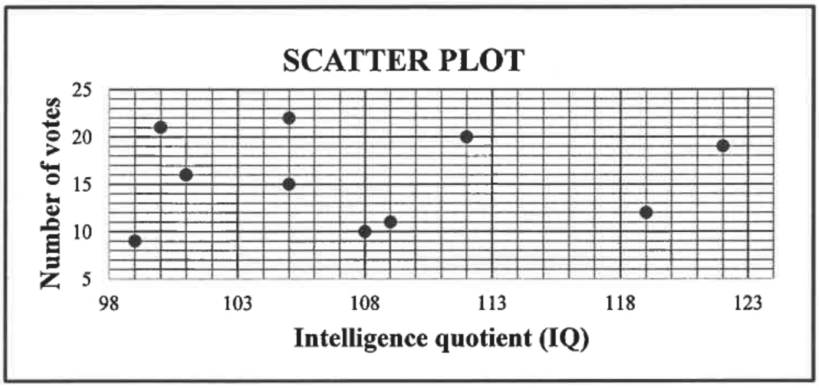
`

(source: maths clinic smart preparation grade 12)

PAPER A

QUESTION 1

The matric class of a certain high school had to vote for the chairperson of the RCL (representative council of learners). The scatter plot below shows the IQ (intelligence quotient) of the 10 learners who received the most votes and the number of votes that they received.



Before the election, the popularity of each of these ten learners was established and a popularity score (out of a 100) was assigned to each. The popularity scores and the number of votes of the same 10 learners who received the most votes are shown in the table below.

Popularity score (x)	32	89	35	82	50	59	81	40	79	65
Number of votes (y)	9	22	10	21	11	15	20	12	19	16

1.1 Calculate the:

1.1.1 Mean number of votes that these 10 learners received (2)

1.1.2 Standard deviation of the number of votes that these 10 learners received (1)

1.2 The learners who received fewer votes than one standard deviation below the mean were not invited for an interview. How many learners were invited? (2)

1.3 Determine the equation of the least squares regression line for the data given in the table. (3)

1.4 Predict the number of votes that a learner with a popularity score of 72 will receive. (2)

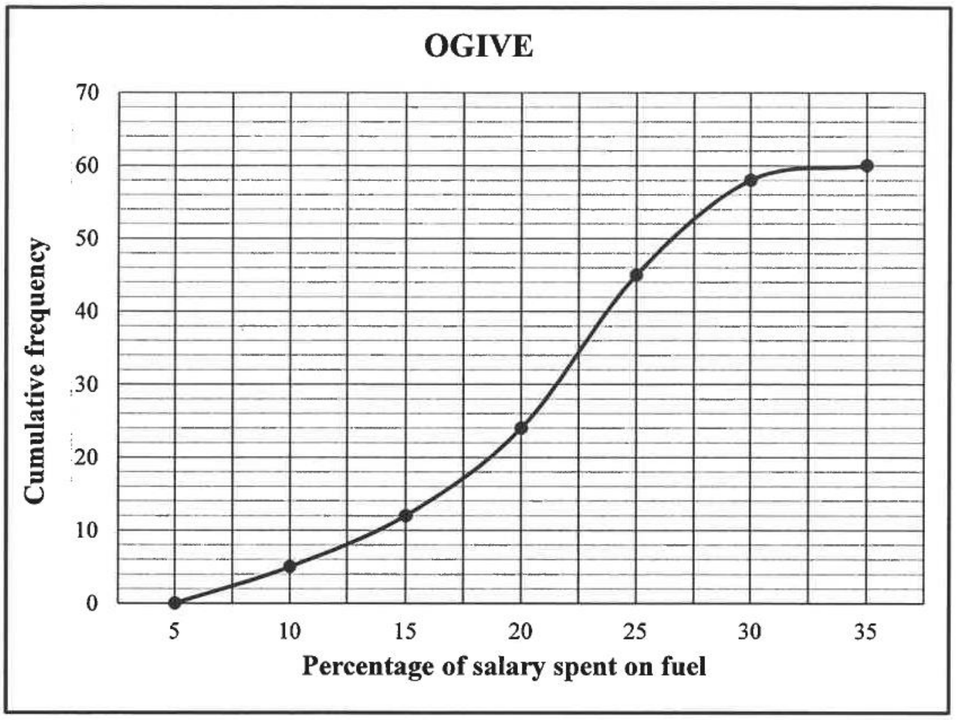
1.5 Using the scatter plot and table above, provide a reason why:

1.5.1 IQ is not a good indicator of the number of votes that a learner could receive (1)

1.5.2 The prediction in QUESTION 1.4 is reliable (1)
[12]

QUESTION 2

A company conducted research among all its employees on what percentage of their monthly salary was spent on fuel in a particular month. The data is represented in the ogive (cumulative frequency graph) below.



2.1 How many people are employed at this company? (1)

2.2 Write down the modal class of the data. (1)

2.3 How many employees spent more than 22,5% of their monthly salary on fuel? (2)

2.4 An employee spent R2 400 of his salary on fuel in that particular month. Determine the monthly salary of this employee if he spends 7% of his salary on fuel. (2)

2.5 The monthly salaries of these employees remains constant and the number of litres of fuel used in each month also remains constant. If the fuel price increases from R21,43 per litre to R22,79 per litre at the beginning of the next month, how will the above ogive change? (2)

[8]

PAPER B

QUESTION 1

Truck drivers travel a certain distance and have a rest before travelling further. A driver kept record of the distance he travelled (in km) on 8 trips and the amount of time he rested (in minutes) before he continued his journey. The information is given in the table below.

Distance travelled (in km) (x)	180	200	400	600	170	350	270	300
Amount of rest time (in minutes) (y)	20	25	55	120	15	50	40	45

1.1 Determine the equation of the least squares regression line for the data. (3)

1.2 If a truck driver travelled 550 km, predict the amount of time (in minutes) that he should rest before continuing his journey. (2)

1.3 Write down the correlation coefficient for the data. (1)

1.4 Interpret your answer to QUESTION 1.3. (1)

1.5 At each stop, the truck driver spent money buying food and other refreshments. The amount spent (in rands) is given in the table below.

100	150	130	200	50	180	200	190

1.5.1 Calculate the mean amount of money he spent at each stop. (2)

1.5.2 Calculate the standard deviation for the data. (1)

1.5.3 At how many stops did the driver spend an amount that was less than one standard deviation below the mean? (2)

[12]

QUESTION 2

At a certain school, the staff committee wanted to determine how many glasses of water the staff members drank during a school day. All teachers present on a specific day were interviewed. The information is shown in the table below.

NUMBER OF GLASSES OF WATER DRANK PER DAY	NUMBER OF STAFF MEMBERS
$0 \leq x < 2$	5
$2 \leq x < 4$	15
$4 \leq x < 6$	13
$6 \leq x < 8$	5
$8 \leq x < 10$	2

2.1 Complete the cumulative frequency column provided in the table in the ANSWER BOOK. (2)

2.2 How many staff members were interviewed? (1)

2.3 How many staff members drank fewer than 6 glasses of water during a school day? (1)

2.4 The staff committee observed that k teachers were absent on the day of the interviews. It was found that half of these k teachers drank from 0 to fewer than 2 (that is $0 \leq x < 2$) glasses of water per day, while the remainder of them drank from 4 to fewer than 6 (that is $4 \leq x < 6$) glasses of water per day. When these k teachers are included in the data, the estimated mean is 4 glasses of water per staff member per day.

How many teachers were absent on the day of the interviews? (4)

[8]

CUMULATIVE FREQUENCE TABLE

2.1

Number of glasses of water drank per day/ *Aantal glase water per dag gedrink*	Number of staff members/ *Getal personeellede*	Cumulative frequency/ *Kumulatiewe frekwensie*
$0 \leq x < 2$	5	
$2 \leq x < 4$	15	
$4 \leq x < 6$	13	
$6 \leq x < 8$	5	
$8 \leq x < 10$	2	

PAPER C

QUESTION 1

The table below shows the mass (in kg) of the school bags of 80 learners.

MASS (in kg)	FREQUENCY
$5 < m \leq 7$	6
$7 < m \leq 9$	18
$9 < m \leq 11$	21
$11 < m \leq 13$	19
$13 < m \leq 15$	11
$15 < m \leq 17$	4
$17 < m \leq 19$	1

1.1 Write down the modal class of the data. (1)

1.2 Complete the cumulative frequency column in the table in the ANSWER BOOK. (2)

1.3 Draw a cumulative frequency graph (ogive) for the given data on the grid provided in the ANSWER BOOK. (3)

1.4 Use the graph to determine the median mass for this data. (2)

1.5 The international guideline for the mass of a school bag is that it should not exceed 10% of a learner's body mass.

1.5.1 Calculate the estimated mean mass of the school bags. (2)

1.5.2 The mean mass of this group of learners was found to be 80 kg. On average, are these school bags satisfying the international guideline with regard to mass? Motivate your answer. (2)

[12]

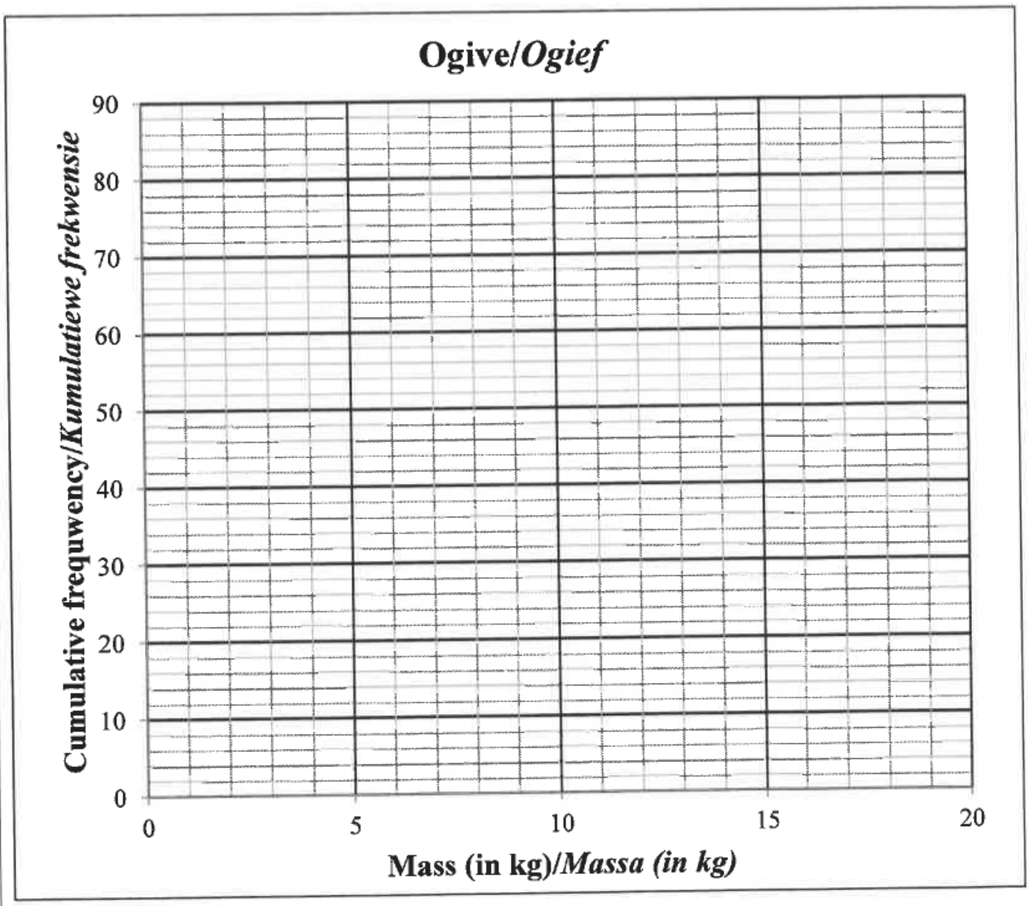
CUMULATIVE FREQUENCE TABLE

1.2

MASS (in kg)/ *MASSA (in kg)*	FREQUENCY/ *FREKWENSIE*	CUMULATIVE FREQUENCY/ *KUMULATIEWE FREKWENSIE*
$5 < m \leq 7$	6	
$7 < m \leq 9$	18	
$9 < m \leq 11$	21	
$11 < m \leq 13$	19	
$13 < m \leq 15$	11	
$15 < m \leq 17$	4	
$17 < m \leq 19$	1	

CUMULATIVE FREQUENCE GRAPH (OGIVE) GRID

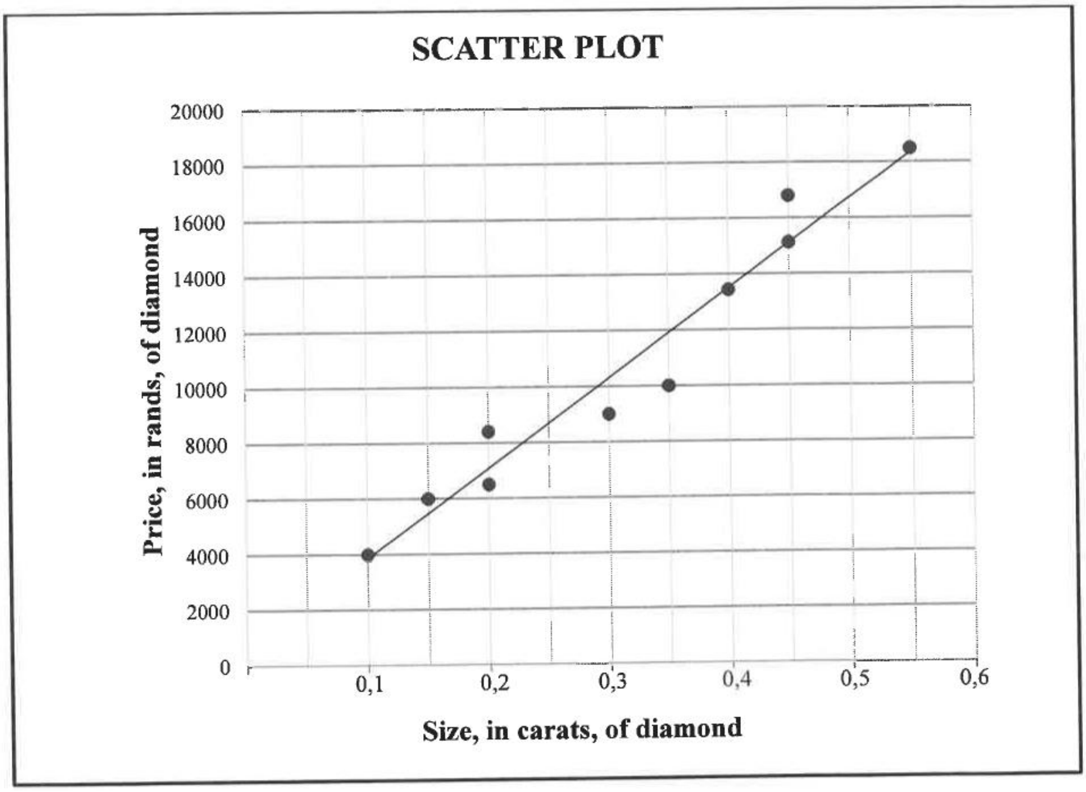
1.3



QUESTION 2

The table below shows the size (in carats) and the price (in rands) of 10 diamonds that were sold by a diamond trader. This information is also presented in the scatter plot below. The least squares regression line for the data is drawn.

Size, in carats, of diamond (x)	0,1	0,15	0,2	0,2	0,3	0,35	0,4	0,45	0,45	0,55
Price, in rands, of diamond (y)	4 000	6 000	6 500	8 400	9 000	10 000	13 440	15 120	16 800	18 480



2.1 Determine the equation of the least squares regression line for the data. (3)

2.2 If the trader sold a diamond that was 0,25 carats in size, predict the selling price of this diamond in rands. (2)

2.3 Calculate the average price increase per 0,05 carat of the diamonds. (2)

2.4 It was later found that the selling price of the 0,35 carat diamond was recorded incorrectly. The correct price is R11 500. When this correction is made to the data set, the correlation between the size and price of these diamonds gets stronger. Explain the reason for this by referring to the given scatter plot. (1)

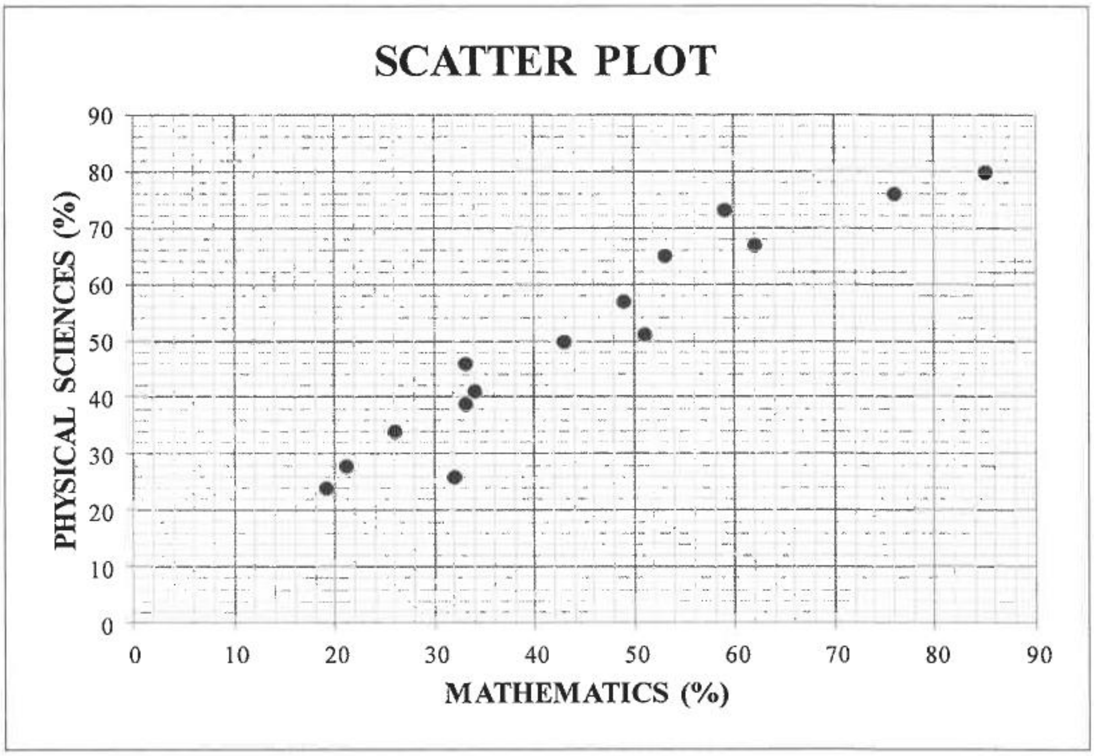
[8]

PAPER D

QUESTION 1

A Mathematics teacher was curious to establish if her learners' Mathematics marks influenced their Physical Sciences marks. In the table below, the Mathematics and Physical Sciences marks of 15 learners in her class are given as percentages (%).

MATHEMATICS (AS %)	26	62	21	33	53	76	32	59	43	33	49	51	19	34	85
PHYSICAL SCIENCES (AS %)	34	67	28	46	65	76	26	73	50	39	57	51	24	41	80



1.1 Determine the equation of the least squares regression line for the data. (3)

1.2 Draw the least squares regression line on the scatter plot provided in the ANSWER BOOK. (2)

1.3 Predict the Physical Sciences mark of a learner who achieved 69% for Mathematics. (2)

1.4 Write down the correlation coefficient between the Mathematics and Physical Sciences marks for the data. (1)

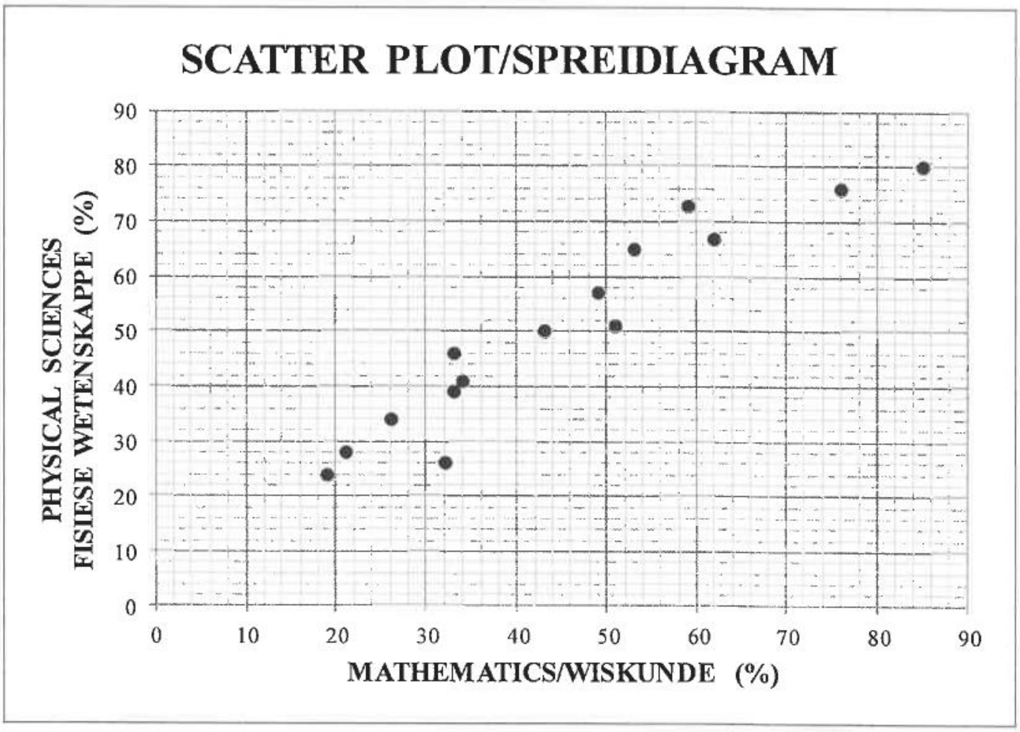
1.5 Comment on the strength of the correlation between the Mathematics and Physical Sciences marks for the data. (1)

1.6 What trend did the teacher observe between the results of the two subjects? (1)

[10]

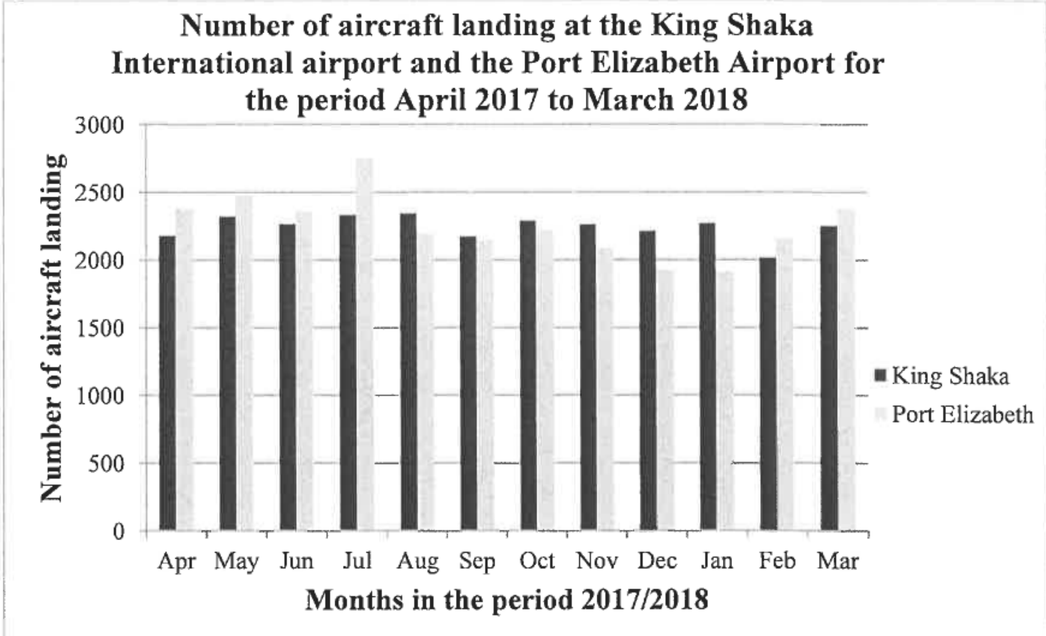
SCATTER PLOT

1.2



QUESTION 2

The number of aircraft landing at the King Shaka International Airport and the Port Elizabeth Airport for the period starting in April 2017 and ending in March 2018, is shown in the double bar graph below.



[Source: http://www.airports.co.za/airports/statistics/aircraft]

2.1 The number of aircraft landing at the Port Elizabeth Airport exceeds the number of aircraft landing at the King Shaka International Airport during some months of the given period. During which month is this difference the greatest? (1)

2.2 The number of aircraft landing at the King Shaka International Airport during these months are:

2 182	2 323	2 267	2 334	2 346	2 175
2 293	2 263	2 215	2 271	2 018	2 254

Calculate the mean for the data. (2)

2.3 Calculate the standard deviation for the number of aircraft landing at the King Shaka International Airport for the given period. (2)

2.4 Determine the number of months in which the number of aircraft landing at the King Shaka International Airport were within one standard deviation of the mean. (3)

2.5 Which ONE of the following statements is CORRECT?

A. During December and January, there were more landings at the Port Elizabeth Airport than at the King Shaka International Airport.

B. There was a greater variation in the number of aircraft landing at the King Shaka International Airport than at the Port Elizabeth Airport for the given period.

C. The standard deviation of the number of landings at the Port Elizabeth Airport will be higher than the standard deviation of the number of landings at the King Shaka International Airport. (1)

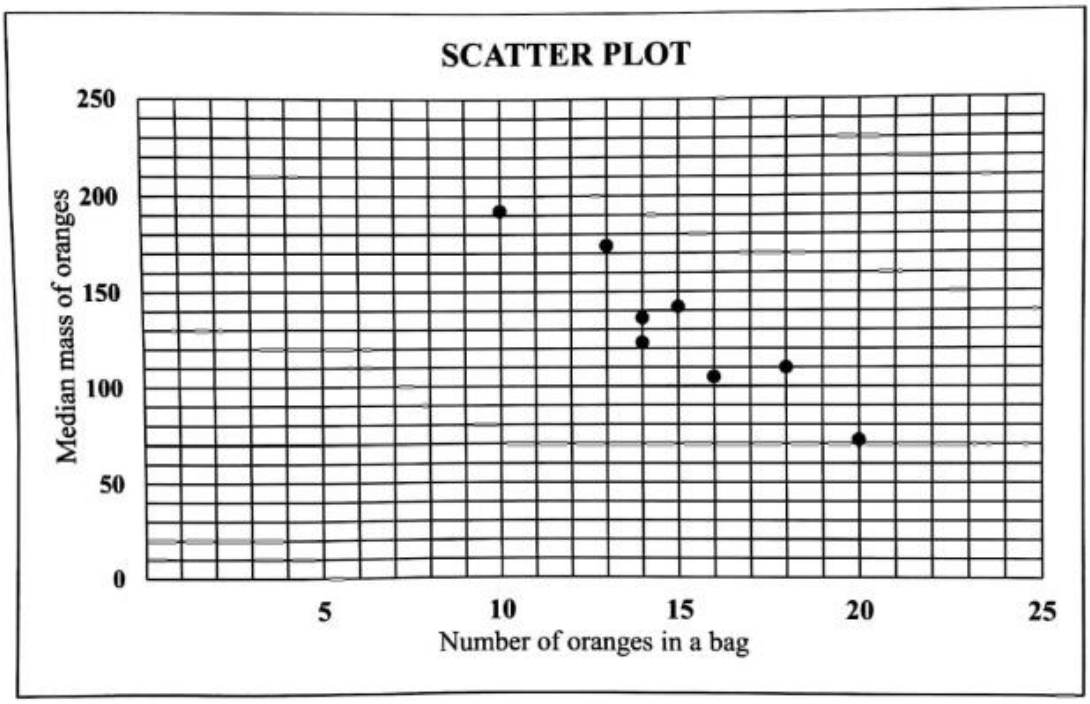
[9]

PAPER E

Question 1

A student is investigating the number of oranges in a bag in relation to the median mass of the oranges filled in the same bag. The findings are recorded in the table below.

Number of oranges in a bag	18	16	20	15	14	13	14	10
Median mass of oranges in the same bag (to the nearest gram)	110	105	72	142	123	174	136	192



1.1 Determine the equation of the least squares regression line for the data. (3)

1.2 Write down the correlation coefficient of the data. (1)

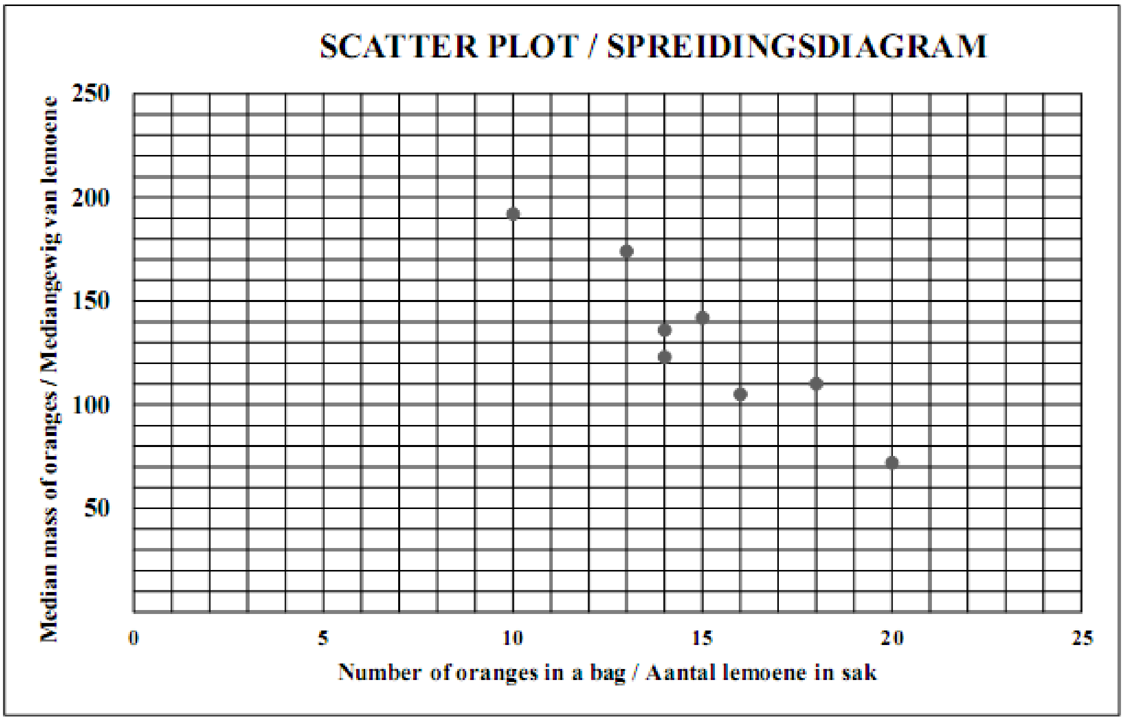
1.3 Draw the least squares regression line on the scatter plot given in your ANSWER BOOK. (2)

1.4 Comment on the strength of the relationship between the number of oranges in the bag and the median mass of the oranges. (1)

1.5 Determine the possible median mass of oranges in a bag, if there are 12 oranges in that bag. (2)

SCATTER PLOT

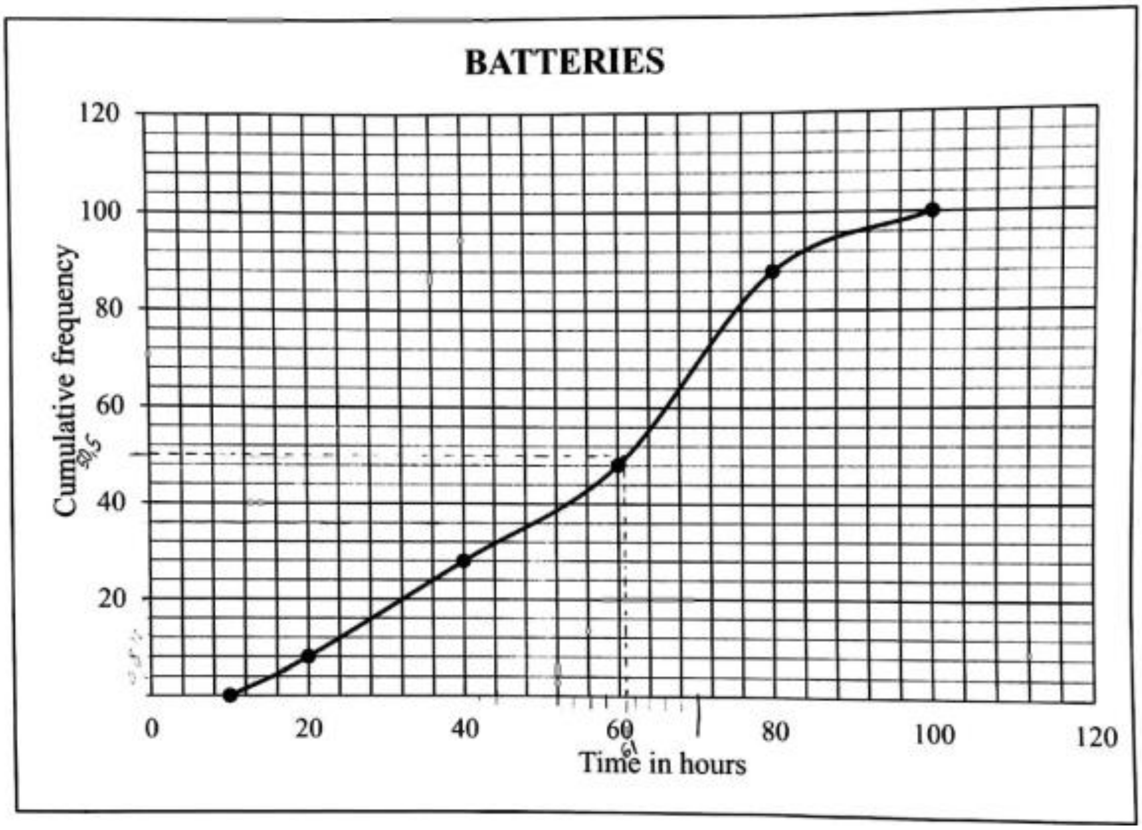
1.3



QUESTION 2

2.1 Batteries are used in everyday life. The Grade 12 Physical Sciences learners investigated the life span of batteries under constant test conditions.

The ogive (cumulative frequency graph) below shows the lifespan (in hours) of the batteries.



2.1.1 How many batteries were tested for this investigation?

2.1.2 Use the graph to estimate the median time for the life span (in hours) of the batteries.

2.1.3 The minimum lifespan of batteries is 10 hours and the maximum lifespan is 100 hours. Use the cumulative frequency graph to draw a box and whisker diagram in your ANSWER BOOK.

2.1.4 Comment on the skewness of the distribution of the lifespan of the batteries.

BOX AND WHISKER NUMBER LINE

2.1.3

2.2 The table below represents values in a data set written in increasing order. None of the values in the data set are repeated.

5	a	19	b	c	d	35

Determine the values of a, b, c, and d if:

- The median is 20.
- The semi interquartile range is 8.
- The upper quartile is twice the lower quartile.
- The mean is 22.

PAPER F

QUESTION 1

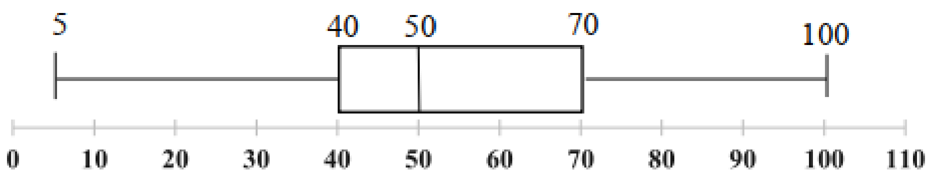
The A-Rithmetic High School decided to compare the results of 31 Grade 12 learners in Mathematics and Physical Sciences in the 2019 Preparatory Examination.

- The Mathematics results are recorded in the table below.
- The box and whisker plot below illustrates the results of Physical Sciences.
- Marks are recorded as percentages.

Mathematics Results

7	11	15	19	19	23	28	28	31	38	39
40	41	48	48	52	53	55	57	59	59	64
67	72	76	83	85	87	89	92	96	-	-

Physical Sciences Results



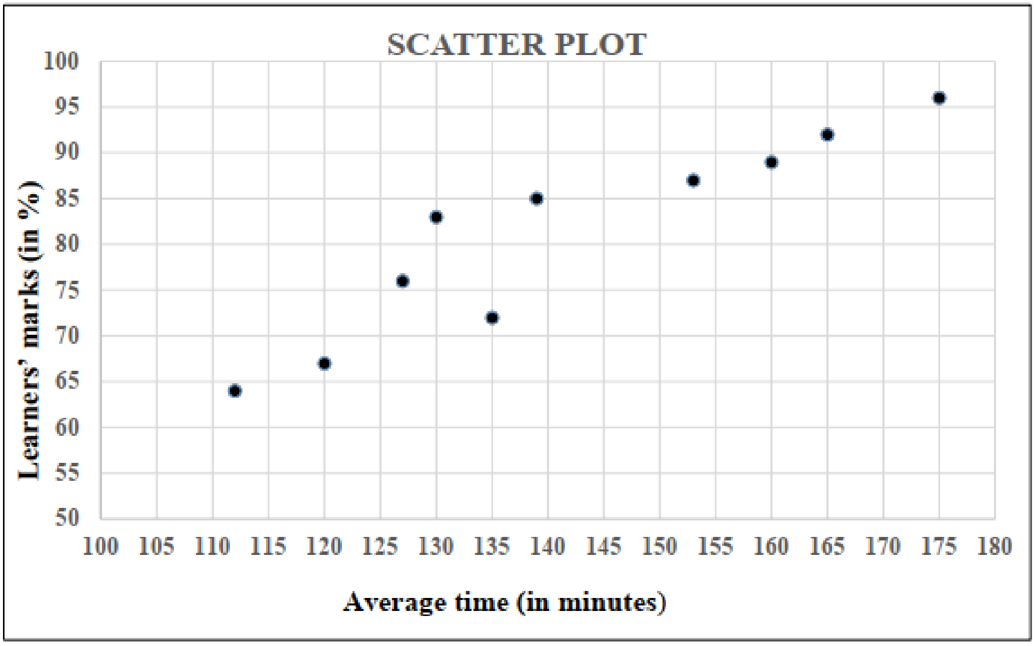
1.1	Calculate the mean mark of the Mathematics learners.	(2)
1.2	Comment on the skewness of the Mathematics data.	(1)
1.3	Determine which subject performed better in the 2019 Preparatory Examination. Give a reason for your conclusion.	(2)
1.4	Write down a possible mark for a learner who achieved the tenth lowest mark in Physical Sciences.	(2)
1.5	A learner scored the fourth highest in both subjects. The learner obtained the GREATEST possible difference between both subjects. Calculate the learner's mark in Physical Sciences.	(2)

QUESTION 2

A question raised by many educators is whether the results that a learner achieves in an examination is dependent on the time that the learner takes to complete the examination.

The average time taken by each of the top 10 Mathematics learners was recorded. The data is represented in the table and scatter plot below.

Average time (in minutes)	175	165	160	153	139	130	127	135	120	112
Learners' marks (in %)	96	92	89	87	85	83	76	72	67	64



2.1 Calculate the equation of the least squares regression line for the data. (3)

2.2 A learner completed the exam in 2,5 hours. Predict the mark that the learner achieved. (2)

2.3 Explain within the context why the regression line is not reliable. (1)

2.4 Calculate the standard deviation of the top 10 Mathematics learners. (2)

2.5 It is further given that $(p\ ;\ 103{,}59)$ is the interval of 15 random learners' marks within ONE standard deviation of the mean. If $\bar{x} = 63{,}96$, calculate the value of p. (3)

PAPER G

QUESTION 1

1.1 Sam recorded the amount of data (in MB) that she had used on each of the first 15 days in April. The information is shown in the table below.

26	13	3	18	12	34	24	58	16	10	15	69	20	17	40

1.1.1 Calculate the:

(a) Mean for the data set (2)

(b) Standard deviation for the data set (1)

1.1.2 Determine the number of days on which the amount of data used was greater than one standard deviation above the mean. (2)

1.1.3 Calculate the maximum total amount of data that Sam must use for the remainder of the month if she wishes for the overall mean of April to be 80% of the mean for the first 15 days. (3)

1.2 The wind speed (in km per hour) and temperature (in °C) for a certain town were recorded at 16:00 for a period of 10 days. The information is shown in the table below.

WIND SPEED IN km/h (x)	2	6	15	20	25	17	11	24	13	22
TEMPERATURE IN °C (y)	28	26	22	22	16	20	24	19	26	19

1.2.1 Determine the equation of the least squares regression line for the data. (3)

1.2.2 Predict the temperature at 16:00 if, on a certain day, the wind speed of this town was 9 km per hour. (2)

1.2.3 Interpret the value of b in the context of the data. (1)

[14]

QUESTION 2

The number of days that Grade 8 learners were absent at a certain high school during a year was recorded. This information is represented in the table below.

NUMBER OF DAYS ABSENT	NUMBER OF LEARNERS
$0 \leq x < 5$	34
$5 \leq x < 10$	45
$10 \leq x < 15$	98
$15 \leq x < 20$	43
$20 \leq x < 25$	7
$25 \leq x < 30$	3

2.1 Write down the modal class for the data. (1)

2.2 How many learners were absent from school for less than 15 days? (1)

2.3 How many Grade 8 learners are at the school? (1)

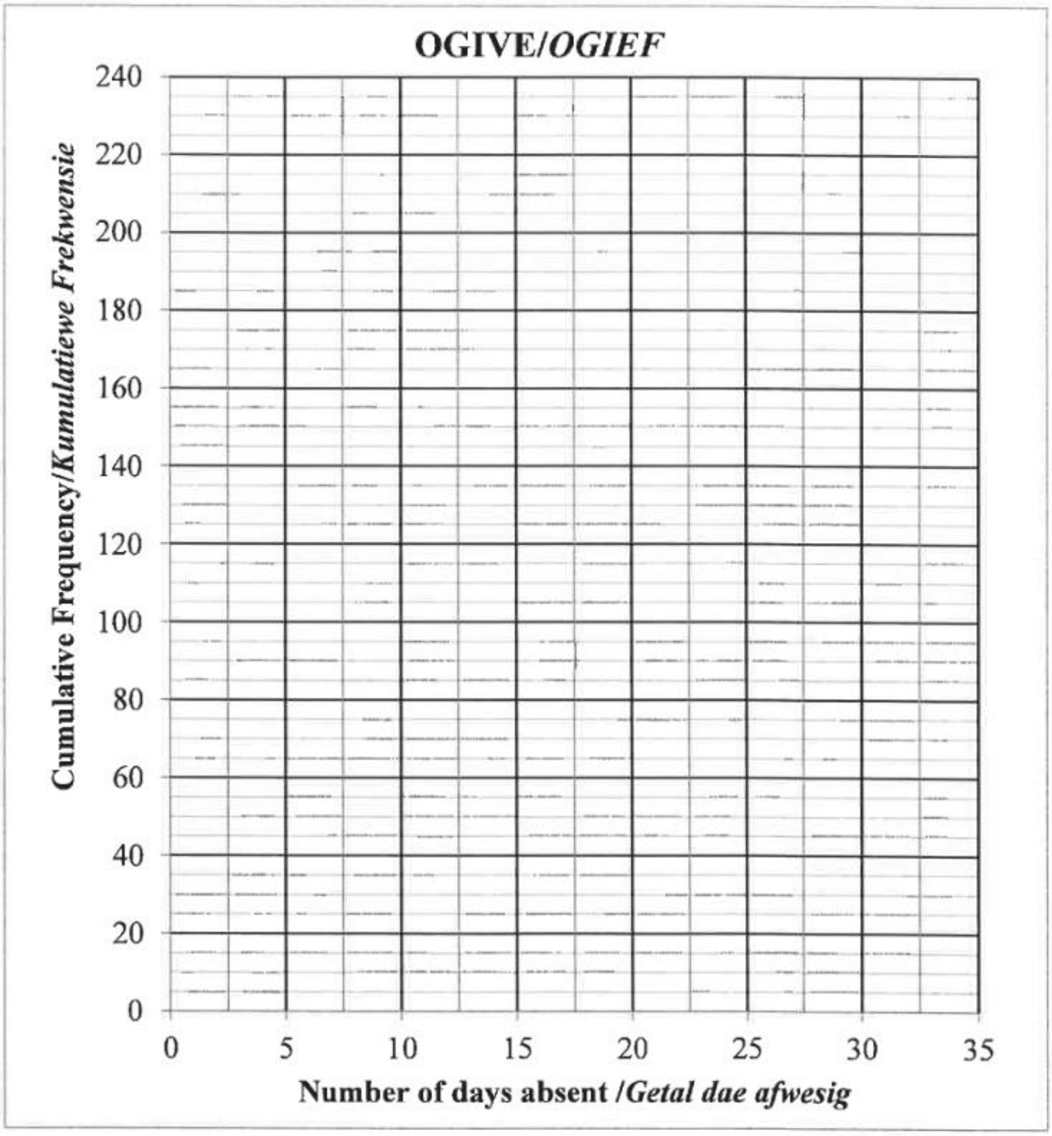
2.4 Draw a cumulative frequency graph (ogive) to represent the data above on the grid provided in the ANSWER BOOK. (4)

2.5 Use the cumulative frequency graph to determine the median number of days the Grade 8 learners were absent. (2)

[9]

CUMULATIVE FREQUENCE GRAPH (OGIVE) GRID

2.4



PAPER H

QUESTION 1

Each child in a group of four-year-old children was given the same puzzle to complete. The time taken (in minutes) by each child to complete the puzzle is shown in the table below.

TIME TAKEN (t) (IN MINUTES)	NUMBER OF CHILDREN
$2 < t \leq 6$	2
$6 < t \leq 10$	10
$10 < t \leq 14$	9
$14 < t \leq 18$	7
$18 < t \leq 22$	8
$22 < t \leq 26$	7
$26 < t \leq 30$	2

1.1 How many children completed the puzzle? (1)

1.2 Calculate the estimated mean time taken to complete the puzzle. (2)

1.3 Complete the cumulative frequency column in the table given in the ANSWER BOOK. (2)

1.4 Draw a cumulative frequency graph (ogive) to represent the data on the grid provided in the ANSWER BOOK. (3)

1.5 Use the graph to determine the median time taken to complete the puzzle. (2)

[10]

CUMULATIVE FREQUENCE TABLE

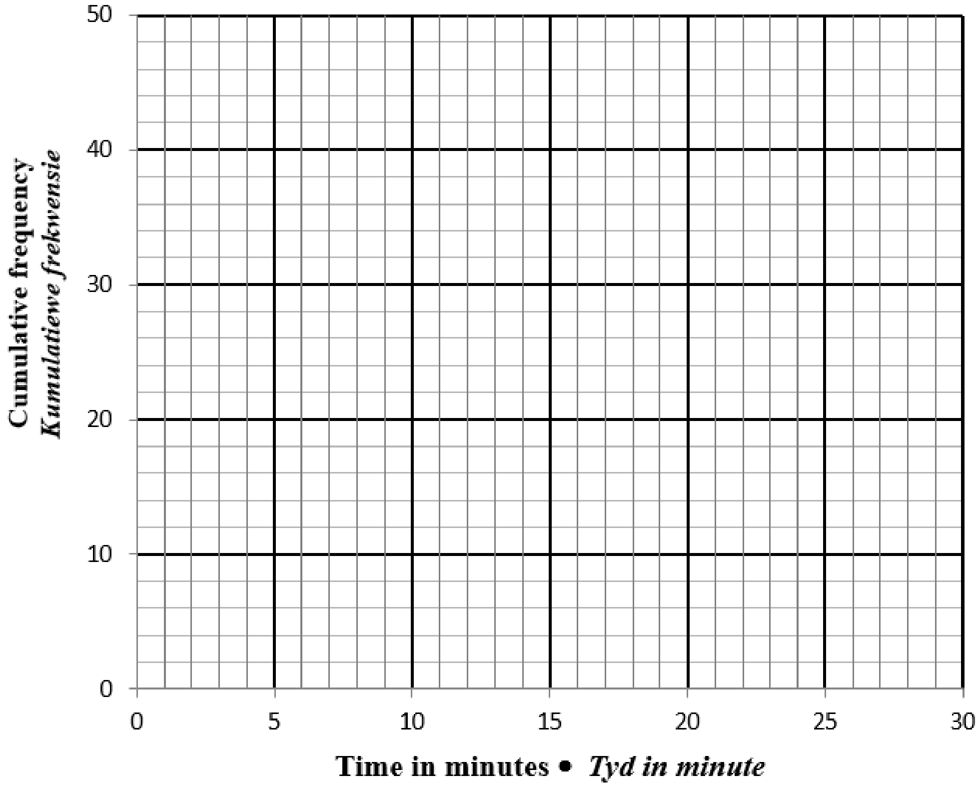
1. 3

Time in minutes (t) *Tyd in minute (t)*	Number of children *Getal kinders*	Cumulative frequency *Kumulatiewe frekwensie*
$2 < t \leq 6$	2	
$6 < t \leq 10$	10	
$10 < t \leq 14$	9	
$14 < t \leq 18$	7	
$18 < t \leq 22$	8	
$22 < t \leq 26$	7	
$26 < t \leq 30$	2	

CUMULATIVE FREQUENCE GRAPH (OGIVE) GRID

1. 4

CUMULATIVE FREQUENCY GRAPH (OGIVE)
KUMULATIEWEFREKWENSIEGRAFIEK (OGIEF)



QUESTION 2

Learners who scored a mark below 50% in a Mathematics test were selected to use a computer-based programme as part of an intervention strategy. On completing the programme, these learners wrote a second test to determine the effectiveness of the intervention strategy. The mark (as a percentage) scored by 15 of these learners in both tests is given in the table below.

LEARNER	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12	L13	L14	L15
TEST 1 (%)	10	18	23	24	27	34	34	36	37	39	40	44	45	48	49
TEST 2 (%)	33	21	32	20	58	43	49	48	41	55	50	45	62	68	60

2.1 Determine the equation of the least squares regression line. (3)

2.2 A learner's mark in the first test was 15 out of a maximum of 50 marks.

2.2.1 Write down the learner's mark for this test as a percentage. (1)

2.2.2 Predict the learner's mark for the second test. Give your answer to the nearest integer. (2)

2.3 For the 15 learners above, the mean mark of the second test is 45,67% and the standard deviation is 13,88%. The teacher discovered that he forgot to add the marks of the last question to the total mark of each of these learners. All the learners scored full marks in the last question. When the marks of the last question are added, the new mean mark is 50,67%.

2.3.1 What is the standard deviation after the marks for the last question are added to each learner's total? (2)

2.3.2 What is the total mark of the last question? (2)

[10]

PAPER I

QUESTION 1

The table below shows the monthly income (in rands) of 6 different people and the amount (in rands) that each person spends on the monthly repayment of a motor vehicle.

MONTHLY INCOME (IN RANDS)	9 000	13 500	15 000	16 500	17 000	20 000
MONTHLY REPAYMENT (IN RANDS)	2 000	3 000	3 500	5 200	5 500	6 000

1.1 Determine the equation of the least squares regression line for the data. (3)

1.2 If a person earns R14 000 per month, predict the monthly repayment that the person could make towards a motor vehicle. (2)

1.3 Determine the correlation coefficient between the monthly income and the monthly repayment of a motor vehicle. (1)

1.4 A person who earns R18 000 per month has to decide whether to spend R9 000 as a monthly repayment of a motor vehicle, or not. If the above information is a true representation of the population data, which of the following would the person most likely decide on:

A Spend R9 000 per month because there is a very strong positive correlation between the amount earned and the monthly repayment.

B NOT to spend R9 000 per month because there is a very weak positive correlation between the amount earned and the monthly repayment.

C Spend R9 000 per month because the point (18 000 ; 9 000) lies very near to the least squares regression line.

D NOT to spend R9 000 per month because the point (18 000 ; 9 000) lies very far from the least squares regression line. (2)

[8]

QUESTION 2

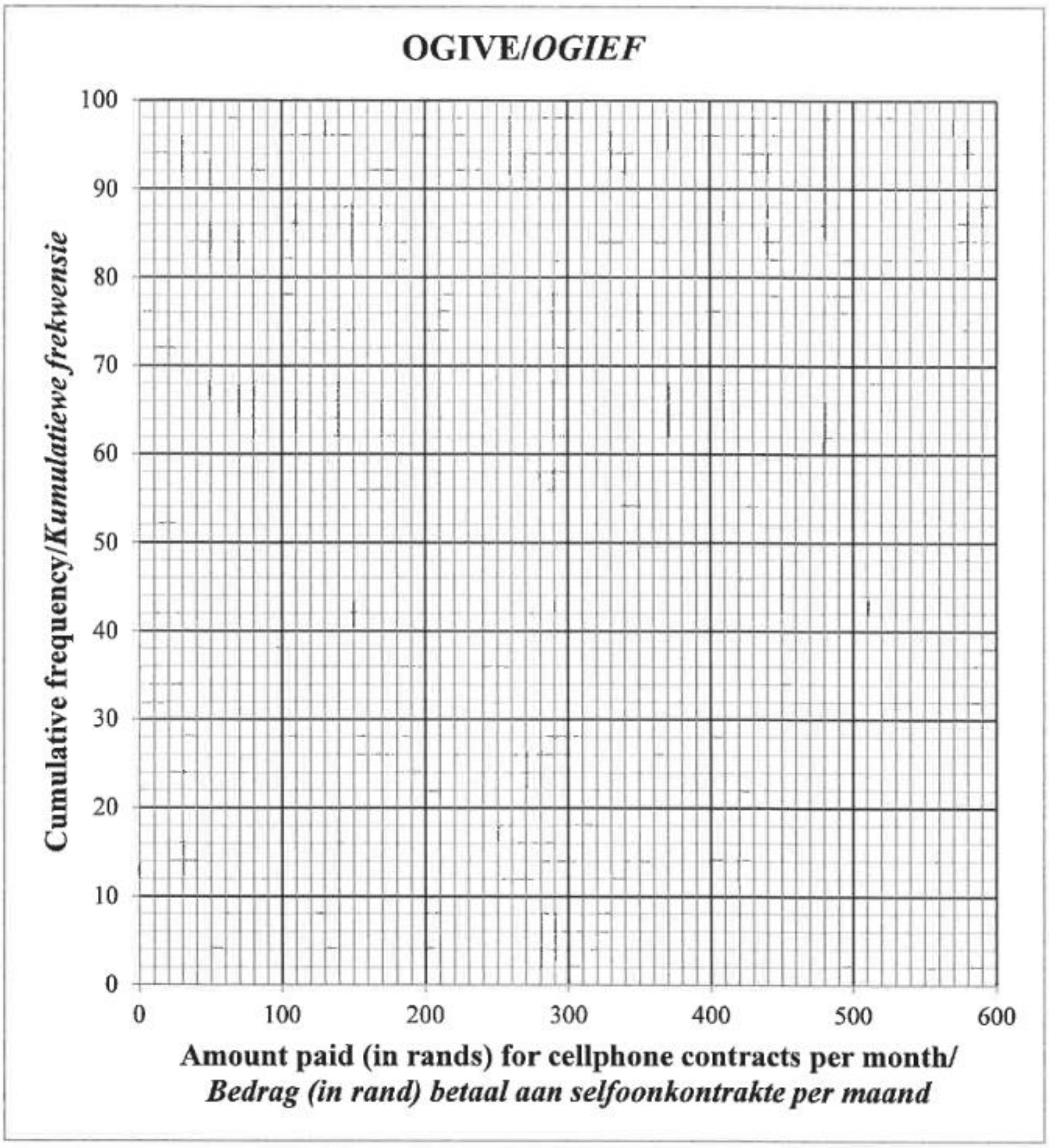
A survey was conducted among 100 people about the amount that they paid on a monthly basis for their cellphone contracts. The person carrying out the survey calculated the estimated mean to be R309 per month. Unfortunately, he lost some of the data thereafter. The partial results of the survey are shown in the frequency table below:

AMOUNT PAID (IN RANDS)	FREQUENCY
$0 < x \leq 100$	7
$100 < x \leq 200$	12
$200 < x \leq 300$	a
$300 < x \leq 400$	35
$400 < x \leq 500$	b
$500 < x \leq 600$	6

2.1	How many people paid R200 or less on their monthly cellphone contracts?	(1)
2.2	Use the information above to show that $a = 24$ and $b = 16$.	(5)
2.3	Write down the modal class for the data.	(1)
2.4	On the grid provided in the ANSWER BOOK, draw an ogive (cumulative frequency graph) to represent the data.	(4)
2.5	Determine how many people paid more than R420 per month for their cellphone contracts.	(2)
		[13]

CUMULATIVE FREQUENCY GRAPH (OGIVE) GRID

2.4



PAPER J

QUESTION 1

1.1 The owner of a small company wishes to establish whether advertising in a regional newspaper is effective. The table below shows the amount spent on advertising and the corresponding sales figures for the last 9 years.

Amount spent on advertising (in rands) (x)	21 300	23 700	24 800	30 540	24 100	40 680	22 400	35 250	29 110
Sales (in rands) (y)	311 500	326 700	349 200	470 000	316 100	564 200	314 000	487 300	392 900

1.1.1 Determine the equation of the least squares regression line for the data. (3)

1.1.2 Predict the sales for a year in which the company will spend R28 500 on advertising. (2)

1.1.3 Write down the correlation coefficient of the data. (1)

1.1.4 Describe the association between the amount spent on advertising in the regional newspaper and the sales of this company. (1)

1.2 The profit that the small company made over the same 9 years is given in the table below.

Profit (in rands)	110 750	107 376	152 338	244 480	144 021	275 994	121 900	207 636	187 700

1.2.1 Calculate the mean profit made over the 9 years. (2)

1.2.2 Write down the standard deviation for the data. (1)

1.2.3 Determine the number of years in which the company made a profit that was greater than one standard deviation above the mean. (2)

[12]

QUESTION 2

The ages of the people who attended a music concert was summarised in the table below.

AGE	NUMBER OF PEOPLE
$5 < x \leq 15$	20
$15 < x \leq 25$	25
$25 < x \leq 35$	60
$35 < x \leq 45$	90
$45 < x \leq 55$	55
$55 < x \leq 65$	40
$65 < x \leq 75$	30

2.1 Write down the modal class of the data. (1)

2.2 How many people attended the music concert? (1)

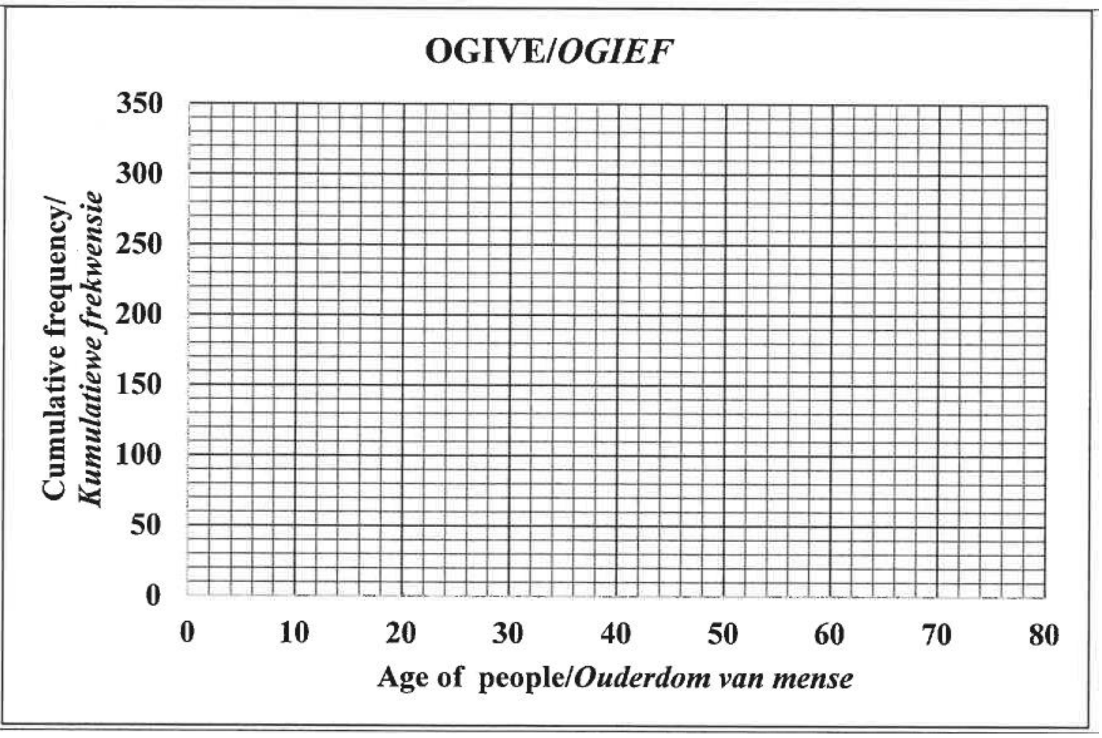
2.3 On the grid provided in the ANSWER BOOK, draw a cumulative frequency graph (ogive) to represent the above data. (4)

2.4 Use the cumulative frequency graph to determine the median age of the people who attended the music concert. (2)

[8]

CUMULATIVE FREQUENCE GRAPH (OGIVE) GRID

2.3


OGIVE/*OGIEF*
Cumulative frequency/
Kumulatiewe frekwensie
350
300
250
200
150
100
50
0
0
10
20
30
40
50
60
70
80
Age of people/*Ouderdom van mense*

ANALYTICAL GEOMETRY

GRADE 11

1.1 The Lines Analysis

The exploration of the formulas used in **Analytical Geometry** to interpret lines and shapes.

Given two points **A** and **B** on the Cartesian plane, with $A(x_A; y_B)$ and $B(x_B; y_B)$.

a) The **Gradient/Slope** of the line **AB** will be given by;

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A}$$

NOTE: If two lines **AB** and **CD** are **Parallel**, then $m_{AB} = m_{BC}$.

And if two lines **AB** and **CD** are **Perpendicular**, then $m_{AB} \times m_{BC} = -1$.

b) The **Midpoint M** of the line **AB** will be given by;

$$\boldsymbol{M\left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right)}$$

c) The **Distance** of the line **AB** between two points **A** and **B** will be given by;

$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

NOTE: If the line **AB** is **Vertical**, then we only consider the $y - values$ of the points, i.e. $d_{AB} = y_{top} - y_{bottom}$

If the line **AB** is **Horizontal**, then we only consider the $x - values$ of the points, $d_{AB} = x_{right} - x_{left}$

d) The $Equation\ of\ the\ straight\ line$ passing through A or B is given by;

$y - y_A = m_{AB}(x - x_A)$ or $y - y_B = m_{AB}(x - x_B)\ or\ y = mx + c$

e) The $Acute\ Angle\ \theta$ made by the line AB with the $x - axis$ on the right side is given by;

$$tan\,\theta = m_{AB}$$

If the Angle made by the line **AB** is **Obtuse**, then we use the concept of **Supplementary Angles** and we **ADD 180^0** to the answer.

1.2 The Shapes Analysis

There is quite a number of shapes that have the concept of straight line geometry, but for this particular level, we will be focusing on the **Triangles** and **Quadrilaterals**. It is also very important to know the properties of all these shapes for a complete application of the concepts and analysis, the determinations and declarations.

The following shapes can/may be analysed using the formulas above and their properties…

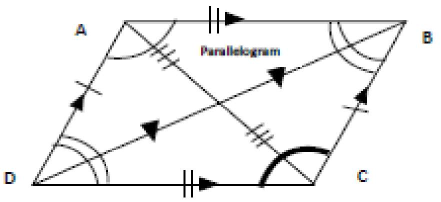
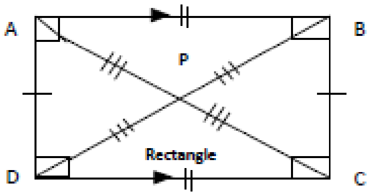
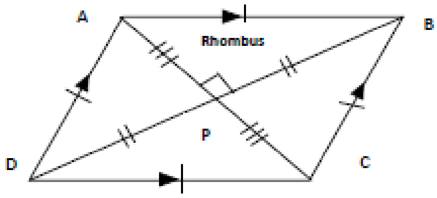
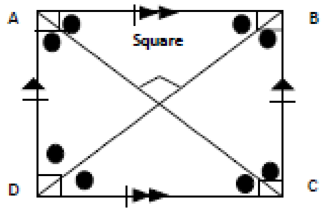
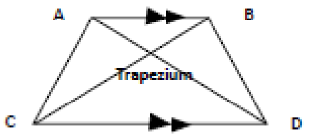
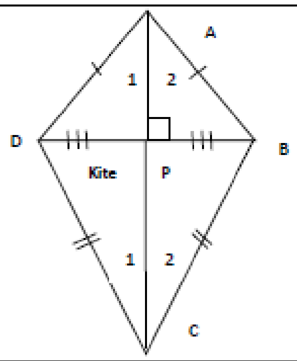
Triangles:

- Right-Angled triangle
- Isosceles triangle
- Equilateral triangle
- Scalene triangle

Quadrilaterals:

- Square
- Rectangle
- Kite
- Parallelogram
- Rhombus
- Trapezium

KNOW ALL PROPERTIES OF QUADRILATERALS BUT BE BYSE TO PARALLELOGRAM.

	1. Opposite sides parallel. 2. Opposite sides equal. 3. Opposite angles are equal. 4. Co-interior angles on same side of a transversal are supplementary. 5. Diagonals bisect each other.
	1. All properties of parallelogram 2. Has 4 right angles. 3. Diagonals are equal.
	1. All properties of parallelograms. 2. Has 4 equal sides, 3. Diagonals bisect opposite angles. 4. Diagonals each other at right angle.
	1. All properties of parallelogram, rectangle, and rhombus 2. 4 equal sides and 4 equal (right) angles.
	1. One pair of parallel sides.
	1. 2 pairs of adjacent sides equal. 2. 1 pair of opposite angles equal 3. The long diagonal bisects the short one at right angle. 4. Diagonals bisect opposite angles. $\hat{A}_1 = \hat{A}_2$ $\hat{C}_1 = \hat{C}_2$

GRADE 12

2.1 The Circles Analysis

The equation of a circle with **centre** $(a; b)$ **and radius** r **is** $(x-a)^2+(y-b)^2=r^2$

The above equation is in **centre form.** Another form of a circle's equation with any centre is $ax^2+by^2+cx+dy+e=0$ which is a **general form.**

By multiplying out the equation in centre-form results in the equation in general form.

Similarly by manipulating the equation in general form by **completing the square** for both x and y results in the equation back to centre form.

Example

Determine the co-ordinates of the centre and the length of the radius for each of the following circles.

a) $(x-5)^2+(y+3)^2=9$

Centre: $(5; -3)$ with $r^2=9$

Therefore; $r=3$

b) $x^2+y^2+2x-6y-6=0$

To find the centre and radius of any circle, rewrite the above equation in the form

$$(x-a)^2+(y-b)^2=r^2$$

- Firstly: Add the additive inverse of 6 and group all terms with $x's$ and $y's$ together.

$$x^2+2x+y^2-6y=6$$

Secondly: **complete the square** for the expression in x and y separately.

$$x^2+2x+\left(2\times\frac{1}{2}\right)^2+y^2-6y+\left(-6\times\frac{1}{2}\right)^2$$
$$=6+\left(2\times\frac{1}{2}\right)^2+\left(-6\times\frac{1}{2}\right)^2 \ (x+1)^2+(y-3)^2=16$$

Centre: $(-1; 3)$ and $r^2=16$, $\therefore r=4$

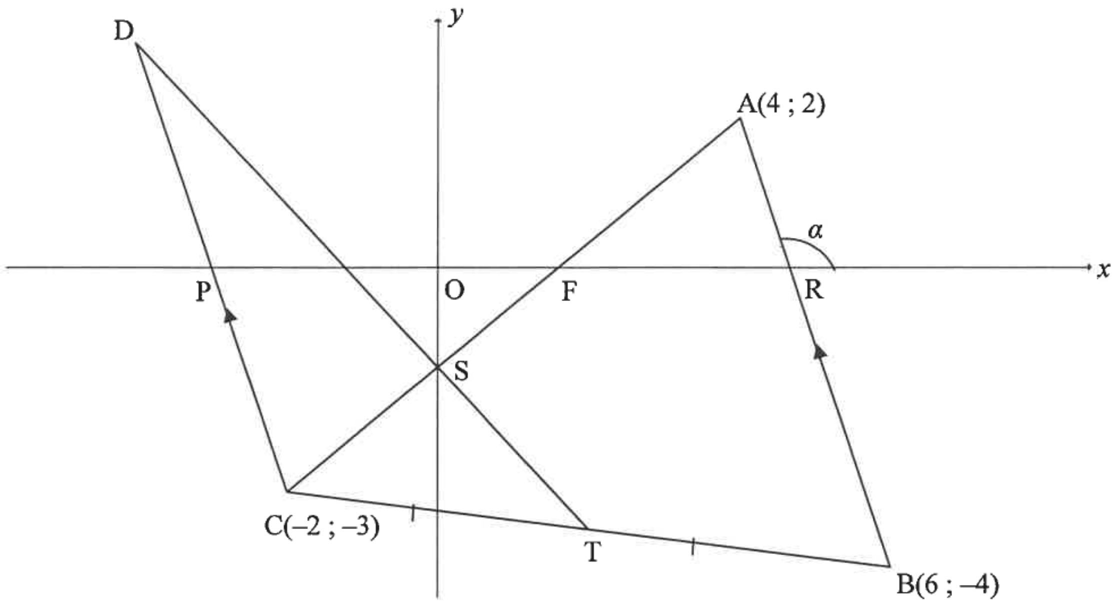
c) Determine the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $A(3; 4)$.

- Draw a sketch
- Calculate the gradient of radius OA.
- Substitute the gradient m_{radius} into $m_{radius} \times m_{tan\,gent} = -1$ to find $m_{tan\,gent}$.
- Use $y - y_A = m_{tan\,gent}(x - x_A)$

PAPER A

QUESTION 3

In the diagram, A(4 ; 2), B(6 ; –4) and C(–2 ; –3) are vertices of ΔABC. T is the midpoint of CB. The equation of line AC is $5x-6y=8$. The angle of inclination of AB is α. ΔDCT is drawn such that CD || BA. The lines AC and DT intersect at S, the y-intercept of AC. P, F and R are the x-intercepts of DC, AC and AB respectively.



3.1 Calculate the:

3.1.1 Gradient of AB (2)

3.1.2 Size of α (2)

3.1.3 Coordinates of T (2)

3.1.4 Coordinates of S (2)

3.2 Determine the equation of CD in the form $y = mx + c$. (3)

3.3 Calculate the:

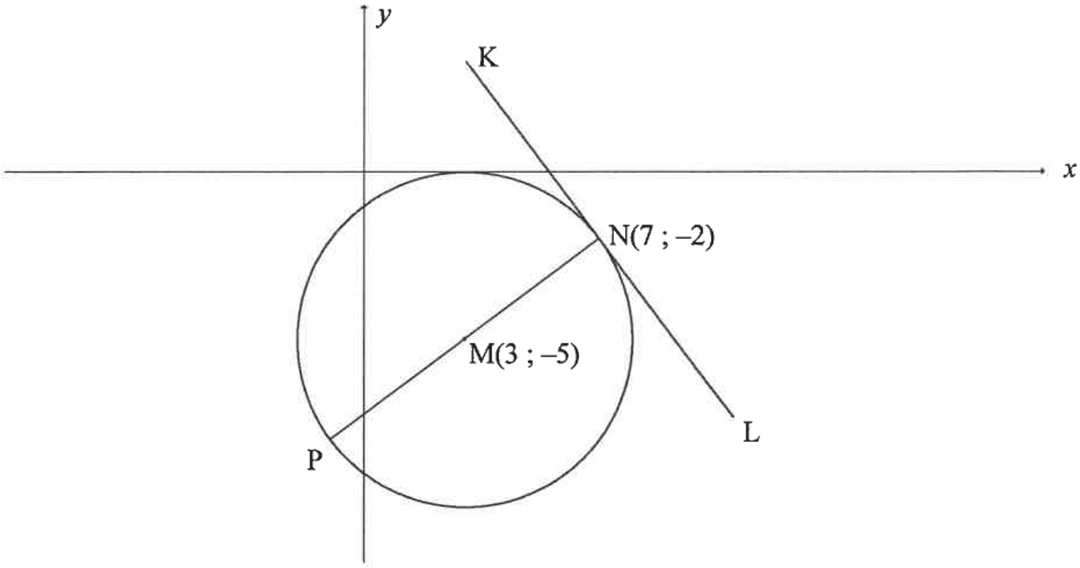
3.3.1 Size of $D\hat{C}A$ (4)

3.3.2 Area of POSC (5)

[20]

QUESTION 4

In the diagram, M(3 ; –5) is the centre of the circle having PN as its diameter. KL is a tangent to the circle at N(7 ; –2).



4.1 Calculate the coordinates of P. (2)

4.2 Determine the equation of:

4.2.1 The circle in the form $(x-a)^2+(y-b)^2=r^2$ (3)

4.2.2 KL in the form $y=mx+c$ (5)

4.3 For which values of k will $y=-\frac{4}{3}x+k$ be a secant to the circle? (4)

4.4 Points A(t ; t) and B are not shown on the diagram.

From point A, another tangent is drawn to touch the circle with centre M at B.

4.4.1 Show that the length of tangent AB is given by $\sqrt{2t^2+4t+9}$. (2)

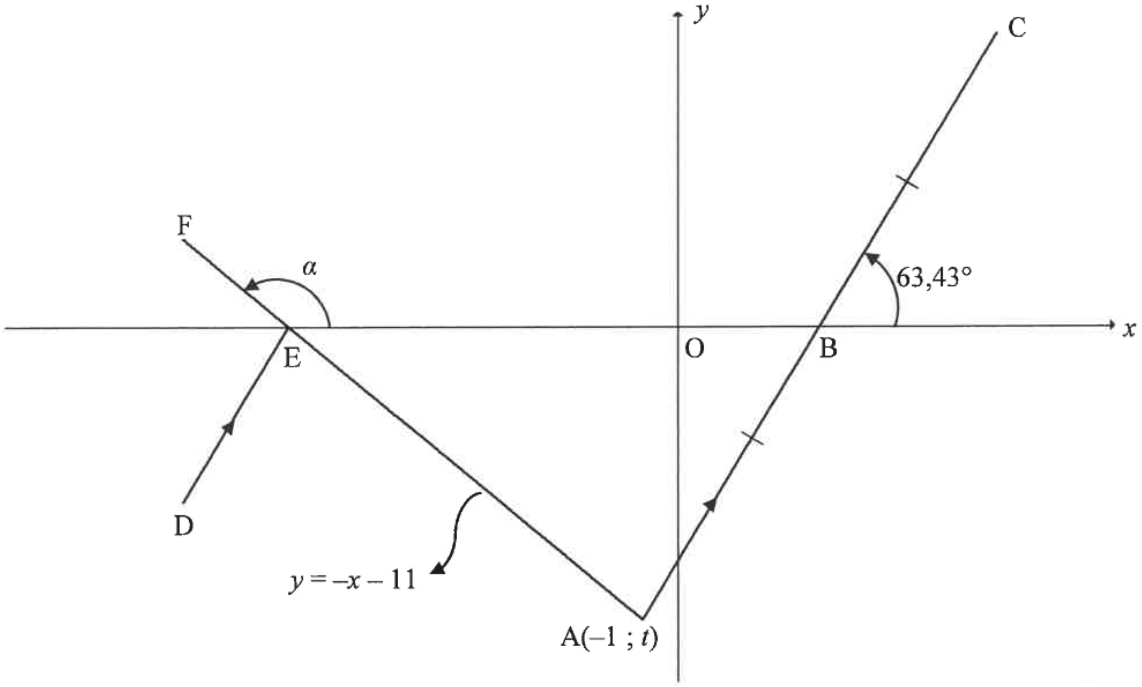
4.4.2 Determine the minimum length of AB. (4)

[20]

PAPER B

QUESTION 3

In the diagram, the equation of line AF is $y = -x - 11$. B, a point on the x-axis, is the midpoint of the straight line joining A$(-1\,;\,t)$ and C. The angles of inclination of AF and AC are α and $63{,}43°$ respectively. AF cuts the x-axis in E. D is a point such that DE || AC.



3.1 Calculate the:

3.1.1 Value of t (2)

3.1.2 Size of α (2)

3.1.3 Gradient of AC, to the nearest whole number (2)

3.2 Determine the equation of AC in the form $y = mx + k$. (2)

3.3 Calculate the:

3.3.1 Coordinates of C (3)

3.3.2 Size of $F\hat{E}D$ (3)

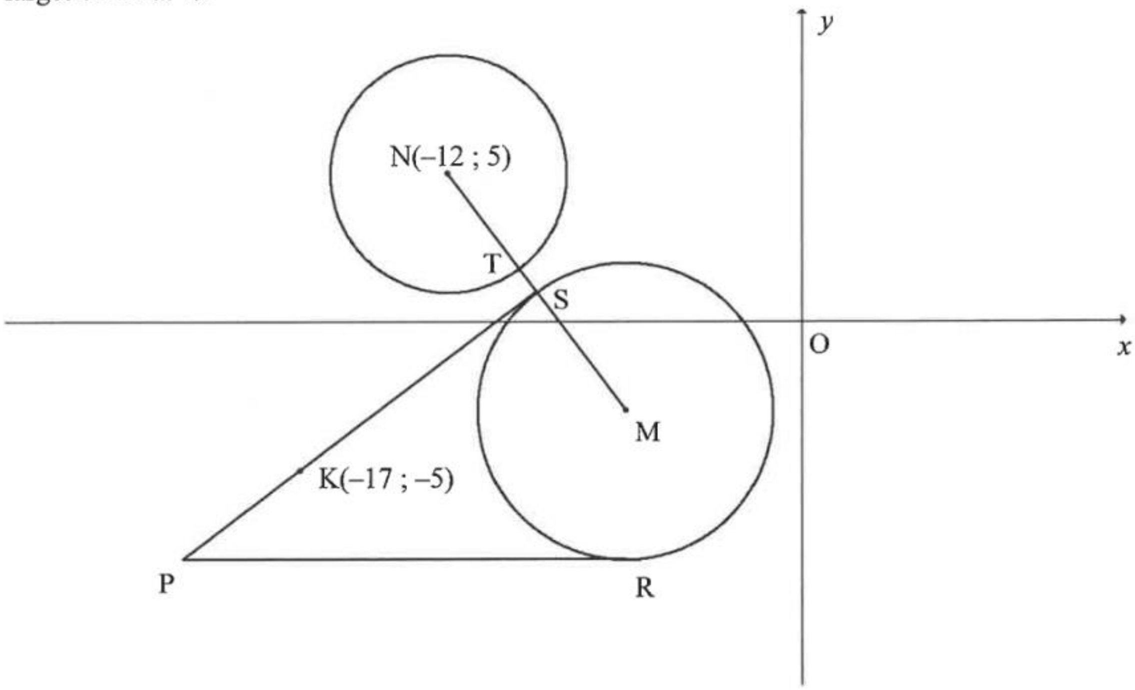
3.4 G is a point such that EAGC, in that order, is a parallelogram.

Determine the equation of a circle centred at G and passing through the point B. Write your answer in the form $(x-a)^2 + (y-b)^2 = r^2$. (4)

[18]

QUESTION 4

In the diagram, the equation of the circle centred at N(–12 ; 5) is $x^2 + y^2 + 24x - 10y + 153 = 0$. The equation of the circle centred at M is $(x+6)^2 + (y+3)^2 = 25$. PS and PR are tangents to the circle centred at M at S and R respectively. PR is parallel to the x-axis. K(–17 ; –5) is a point on PS. The straight line joining N and M cuts the smaller circle at T and the larger circle at S.



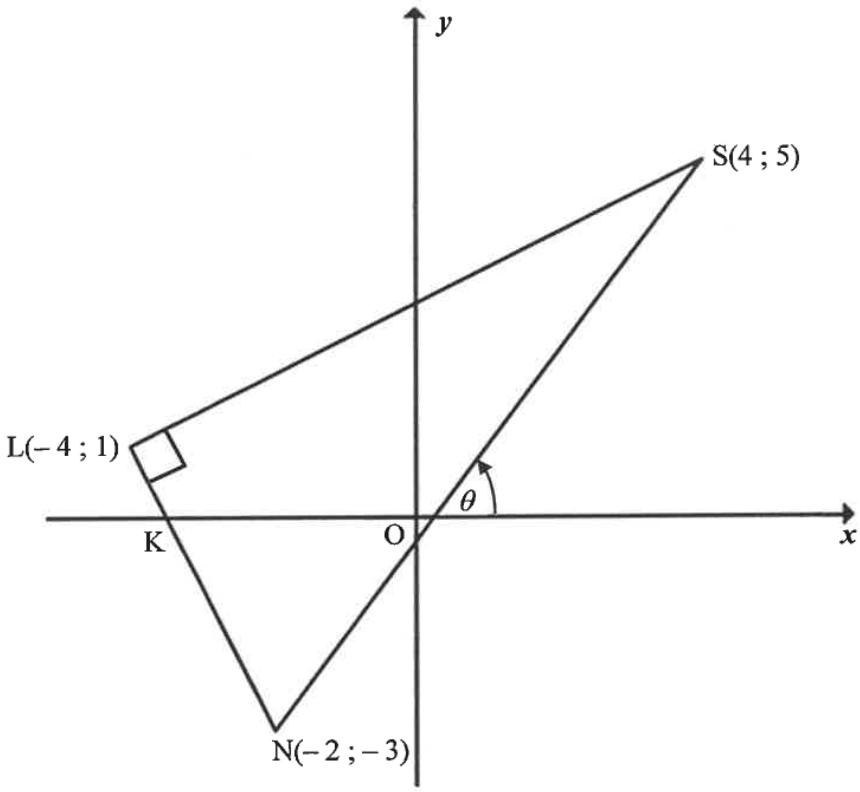
4.1	Write down the coordinates of M.	(2)
4.2	Calculate the:	
	4.2.1 Length of the radius of the smaller circle	(2)
	4.2.2 Length of TS	(4)
4.3	Determine the equation of the tangent:	
	4.3.1 PR	(2)
	4.3.2 PS, in the form $y = mx + c$	(5)
4.4	Quadrilateral PSMR is drawn. Calculate the:	
	4.4.1 Perimeter of PSMR	(5)
	4.4.2 Ratio of $\dfrac{\text{area of } \Delta NPS}{\text{area of quadrilateral PSMR}}$	(2)

[22]

PAPER C

QUESTION 3

In the figure, L$(-4\ ;\ 1)$, S$(4\ ;\ 5)$ and N$(-2\ ;\ -3)$ are the vertices of a triangle having $S\hat{L}N = 90°$. LN intersects the x-axis at K.

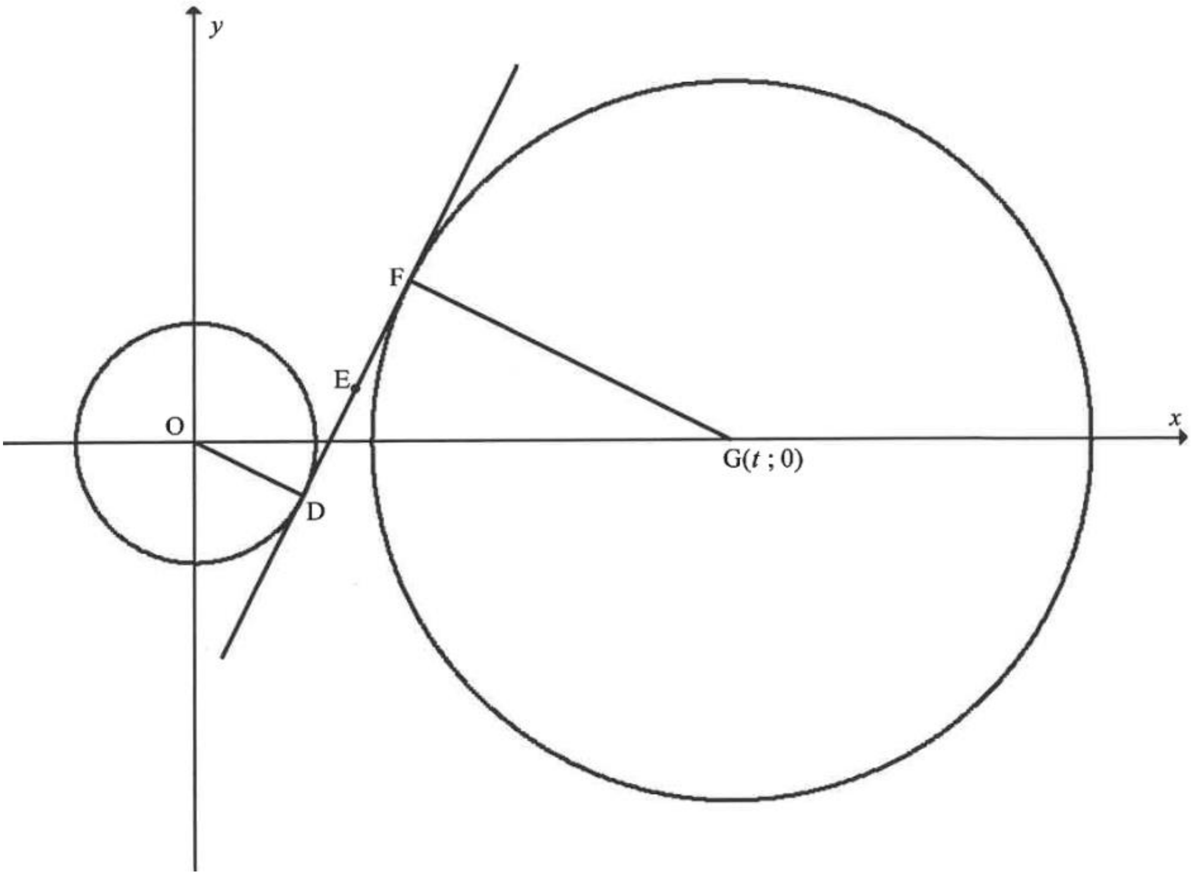


3.1	Calculate the length of SL. Leave your answer in surd form.	(2)
3.2	Calculate the gradient of SN.	(2)
3.3	Calculate the size of θ, the angle of inclination of SN.	(2)
3.4	Calculate the size of $L\hat{N}S$.	(3)
3.5	Determine the equation of the line which passes through L and is parallel to SN. Write your answer in the form $y = mx + c$.	(3)
3.6	Calculate the area of ΔLSN.	(3)
3.7	Calculate the coordinates of point P, which is equidistant from L, S and N.	(3)
3.8	Calculate the size of $L\hat{P}S$.	(2)
		[20]

-

QUESTION 4

In the diagram, the circle with centre O has the equation $x^2 + y^2 = 20$. $G(t\ ;\ 0)$ is the centre of the larger circle. A common tangent touches the circles at D and F respectively, such that $D(p\ ;-2)$ lies in the 4th quadrant.

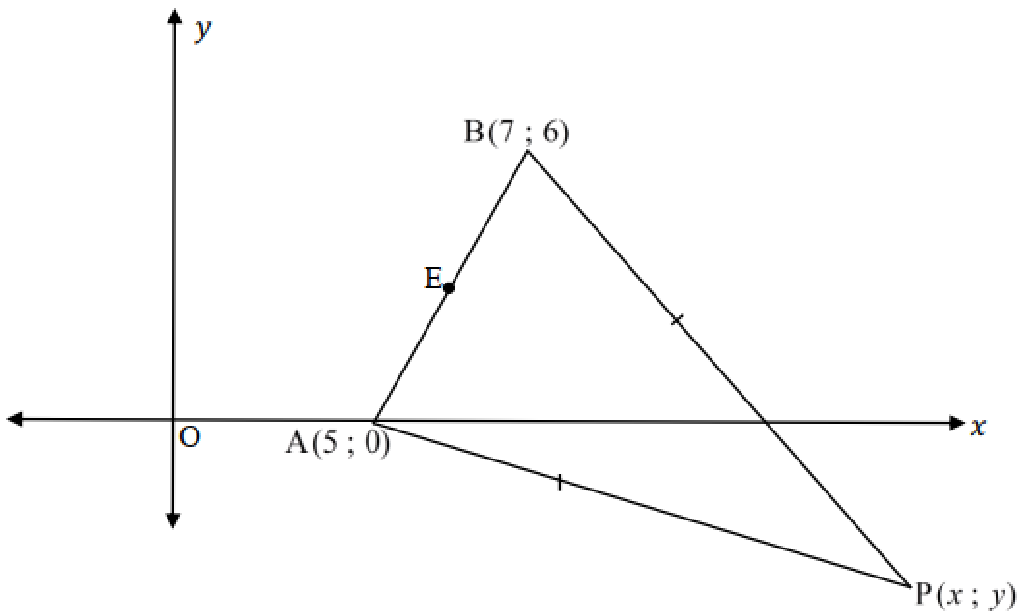


4.1	Given that $D(p\ ;-2)$ lies on the smaller circle, show that $p = 4$.	(2)
4.2	$E(6\ ;\ 2)$ is the midpoint of DF. Determine the coordinates of F.	(3)
4.3	Determine the equation of the common tangent, DF, in the form $y = mx + c$.	(4)
4.4	Calculate the value of t. Show ALL working.	(3)
4.5	Determine the equation of the larger circle in the form $ax^2 + by^2 + cx + dy + e = 0$.	(4)
4.6	The smaller circle must be translated by k units along the x-axis to touch the larger circle internally. Calculate the possible values of k.	(4)
		[20]

PAPER D

QUESTION 3

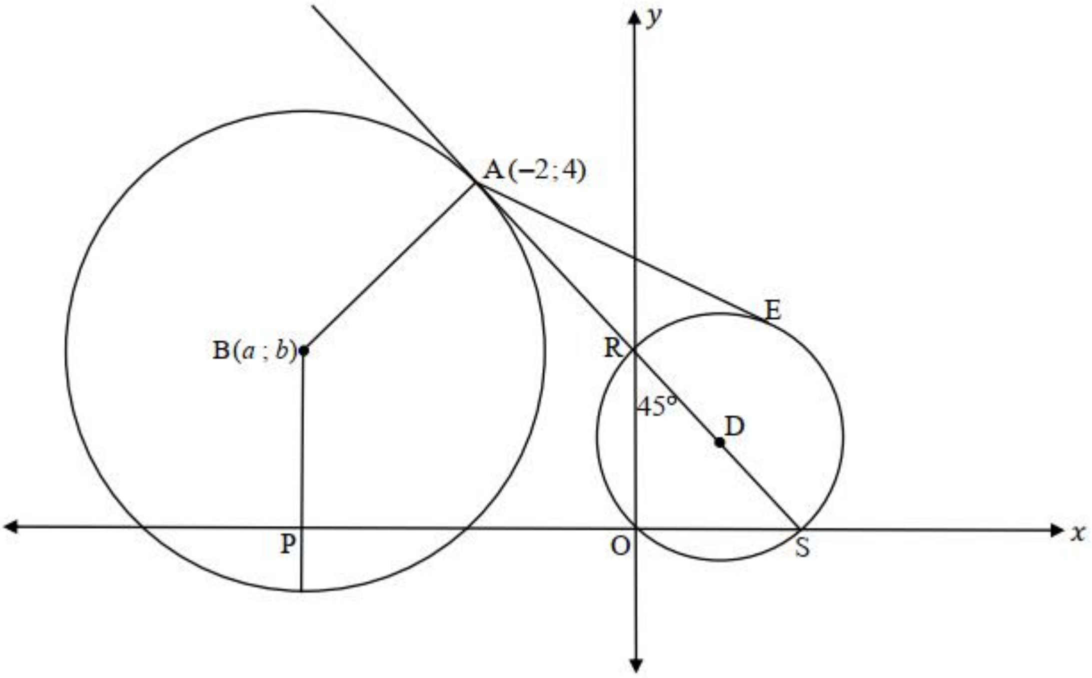
In the diagram below, points $A(5\ ;\ 0)$, $B(7\ ;\ 6)$ and $P(x\ ;\ y)$ form a triangle.
$BP = AP$ and E is the midpoint of AB.



3.1	Determine the coordinates of E.	(2)
3.2	Determine the equation of line BA.	(3)
3.3	Line BA is parallel to the straight line with equation $rx - 3y + 5 = 0$. Calculate the value of r.	(3)
3.4	If the area of $\Delta AOP = 10$ units^2 and $y < 0$, calculate the coordinates of P.	(7)

QUESTION 4

The diagram below shows a circle with centre $B(a\,;b)$. BP is parallel to the y-axis with P on the x-axis. AS is a tangent to circle B at $A(-2\,;4)$ and intersects the x-axis at S and the y-axis at R. AE is a tangent to the smaller circle with centre D and touches the circle at E. $O\hat{R}S = 45°$.

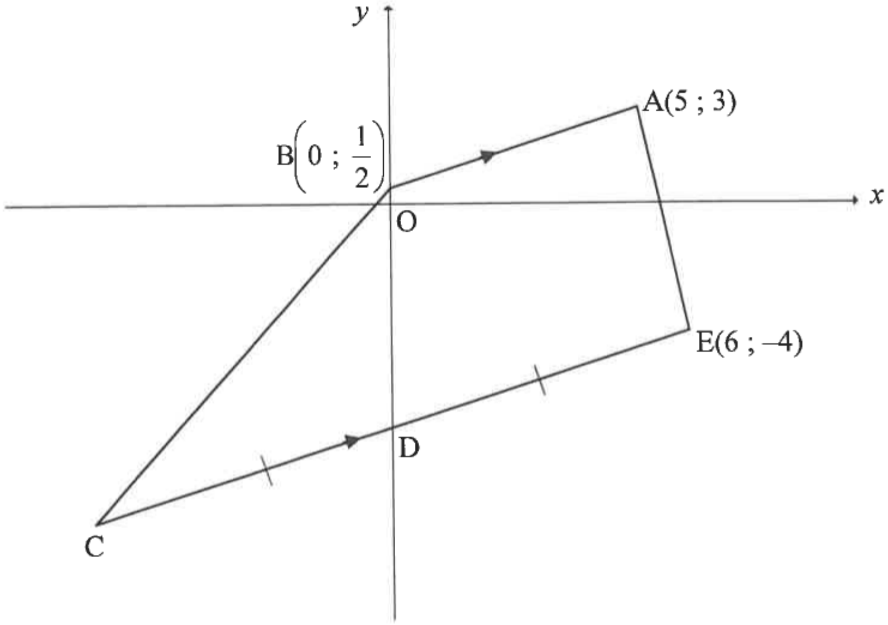


4.1	Determine the equation of tangent AS.	(4)
4.2	If OP = 4 units, determine the values of a and b, the centre of the larger circle.	(4)
4.3	Determine the equation of the circle with centre B.	(3)
4.4	The equation of the smaller circle with centre D is $x^2 - 2x + y^2 - 2y = 0$. Write this equation in the form $(x-a)^2 + (y-b)^2 = r^2$.	(3)
4.5	Write down the coordinates of D, the centre of the smaller circle.	(1)
4.6	Calculate the length of AE, the tangent to circle D at E.	(6)

PAPER E

QUESTION 3

In the diagram, A(5 ; 3), $B\left(0\ ;\ \frac{1}{2}\right)$, C and E(6 ; –4) are the vertices of a trapezium having BA || CE. D is the *y*-intercept of CE and CD = DE.



3.1 Calculate the gradient of AB. (2)

3.2 Determine the equation of CE in the form $y = mx + c$. (3)

3.3 Calculate the:

3.3.1 Coordinates of C (3)

3.3.2 Area of quadrilateral ABCD (4)

3.4 If point K is the reflection of E in the *y*-axis:

3.4.1 Write down the coordinates of K (2)

3.4.2 Calculate the:

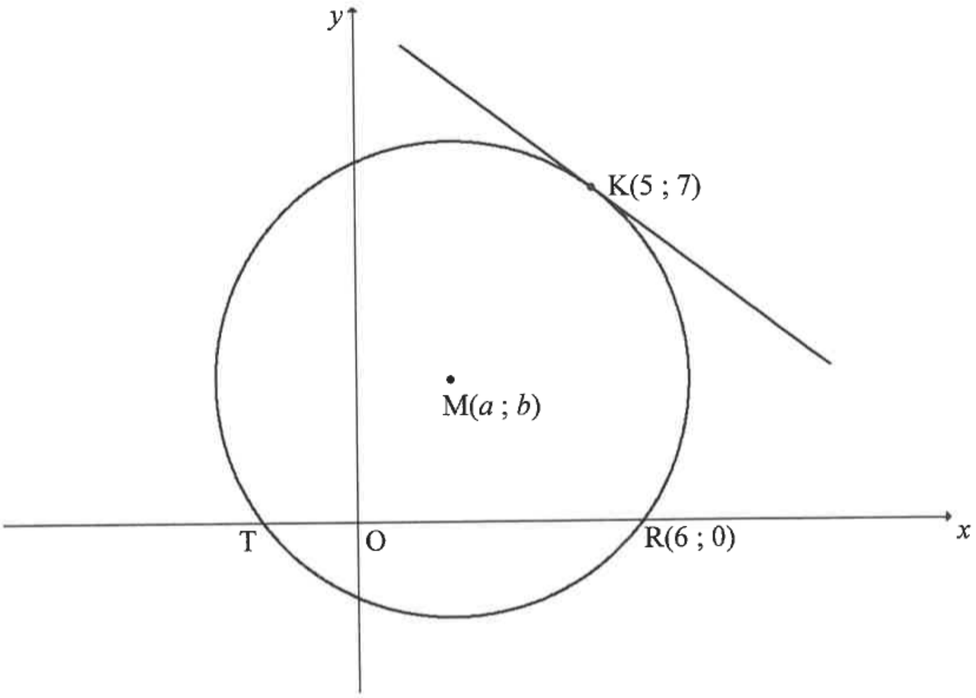
(a) Perimeter of ΔKEC (4)

(b) Size of $K\hat{C}E$ (3)

[21]

QUESTION 4

In the diagram, the circle centred at $M(a\,;b)$ is drawn. T and $R(6\,;0)$ are the x-intercepts of the circle. A tangent is drawn to the circle at $K(5\,;7)$.



4.1 M is a point on the line $y = x + 1$.

4.1.1 Write b in terms of a. (1)

4.1.2 Calculate the coordinates of M. (5)

4.2 If the coordinates of M are $(2\,;3)$, calculate the length of:

4.2.1 The radius of the circle (2)

4.2.2 TR (2)

4.3 Determine the equation of the tangent to the circle at K. Write your answer in the form $y = mx + c$. (5)

4.4 A horizontal line is drawn as a tangent to the circle M at the point $N(c\,;d)$, where $d < 0$.

4.4.1 Write down the coordinates of N. (2)

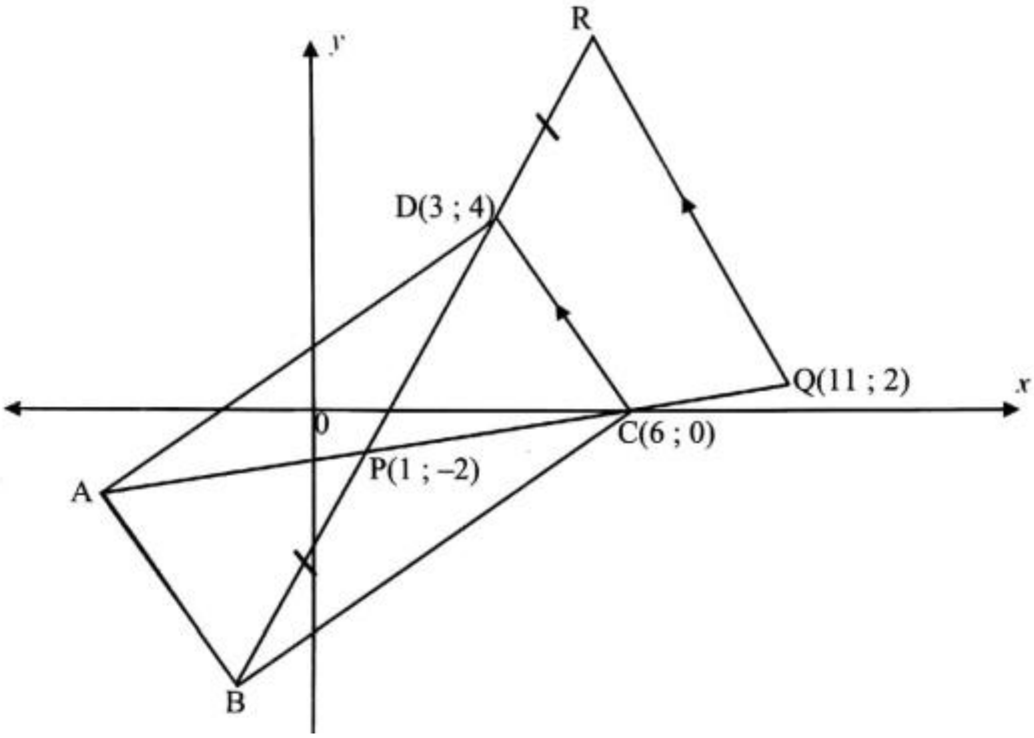
4.4.2 Determine the equation of the circle centred at N and passing through T. Write your answer in the form $(x-a)^2 + (y-b)^2 = r^2$. (3)

[20]

PAPER F

QUESTION 3

3.1 In the diagram below, A, B, C(6 ; 0) and D(3 ; 4) are the vertices of rectangle ABCD. Diagonals AC and BD bisect each other at P(1 ; −2). AC is produced to Q(11 ; 2) and BD is produced to R such that BP = DR and CD ∥ QR.



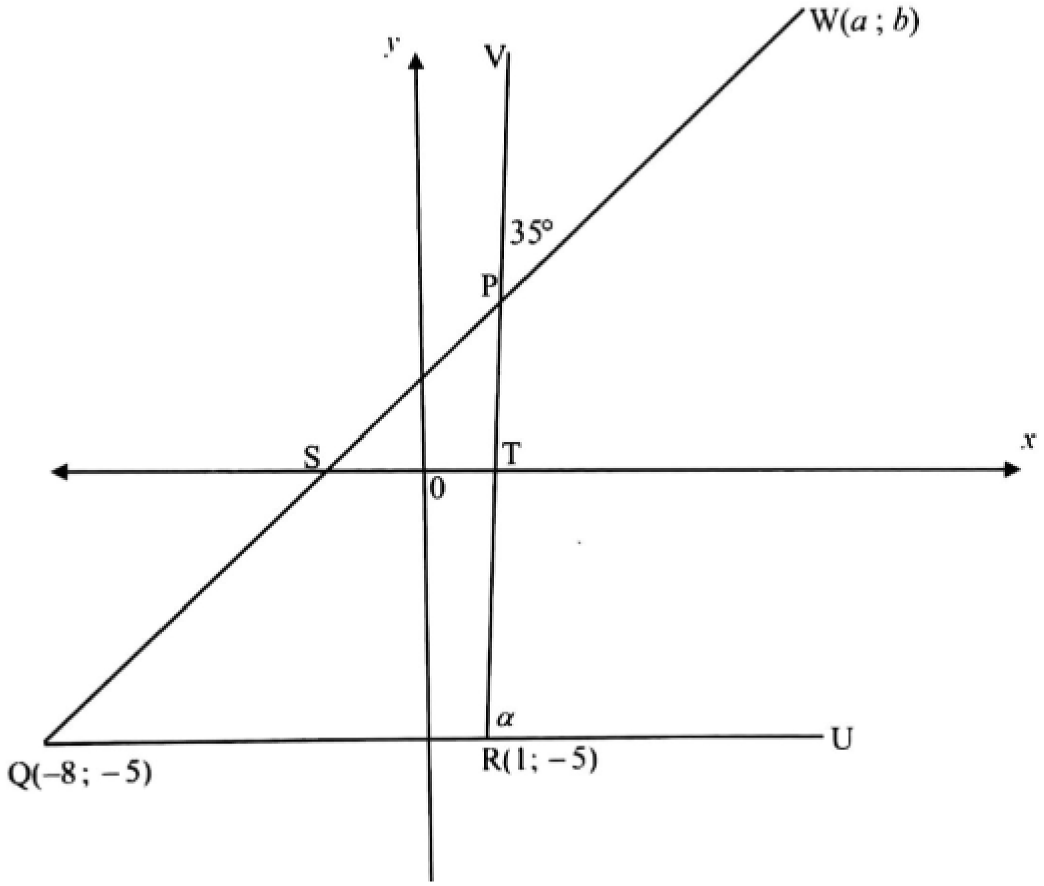
3.1.1 Calculate the coordinates of B. (3)

3.1.2 Determine the gradient of CD. (2)

3.1.3 Show that the equation of QR is $y = -\frac{4}{3}x + \frac{50}{3}$. (2)

3.1.4 If K(4 ; y) is a point in the 4[th] quadrant such that PK = RQ, calculate the value of y. (6)

3.2 In the diagram below, P, Q$(-8\,;-5)$ and R$(1\,;-5)$ are the vertices of ΔPQR. RP is produced to V and QP is produced to W$(a\,;b)$ such that $V\hat{P}W = 35°$. The equation of QW is $y = x + \frac{2}{3}$. QR is produced to U and $U\hat{R}V = \alpha$. QW and RV intersect the x-axis at S and T respectively.



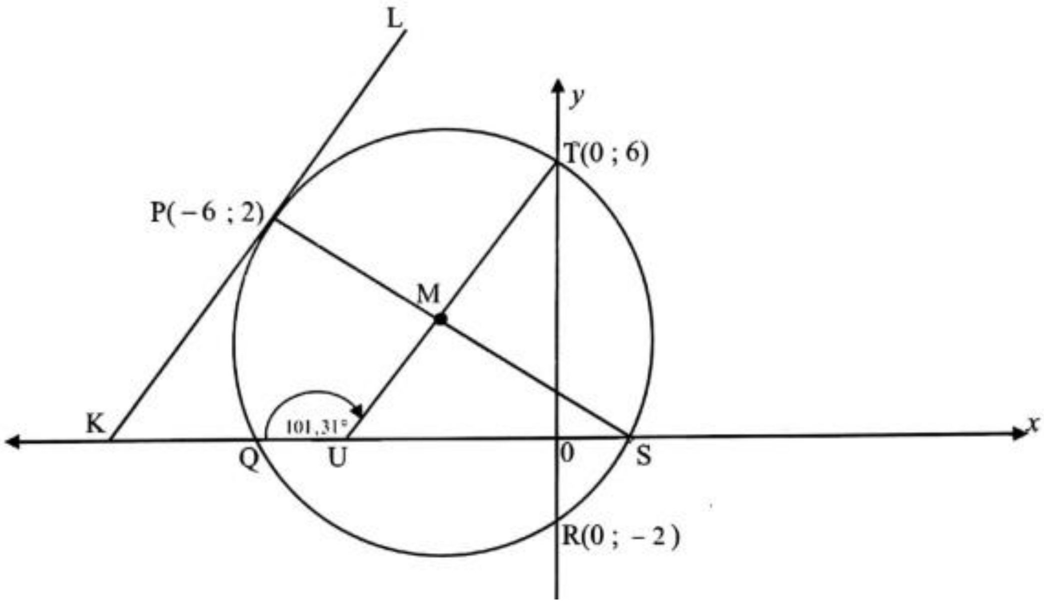
3.2.1 Calculate the size of α. (5)

3.2.2 It is further given that $QU \perp WU$ and R is the midpoint of QU.
Calculate the area of ΔQWU. (6)

QUESTION 4

In the diagram below, a circle with centre M, cuts the x-axis at Q and S and the y-axis at T(0 ; 6) and R(0 ; –2). The equation of diameter SMP is $y=\frac{-1}{5}x+\frac{4}{5}$.

KPL is a tangent to the circle at P(–6 ; 2). TM produced cuts the x-axis at U. $Q\hat{U}T = 101,31°$.



4.1 Determine the equation of TU. (3)

4.2 Calculate the coordinates of M. (3)

4.3 If the coordinates of M are (–1 ; 1) , determine the equation of the circle in the form $(x-a)^2+(y-b)^2=r^2$. (3)

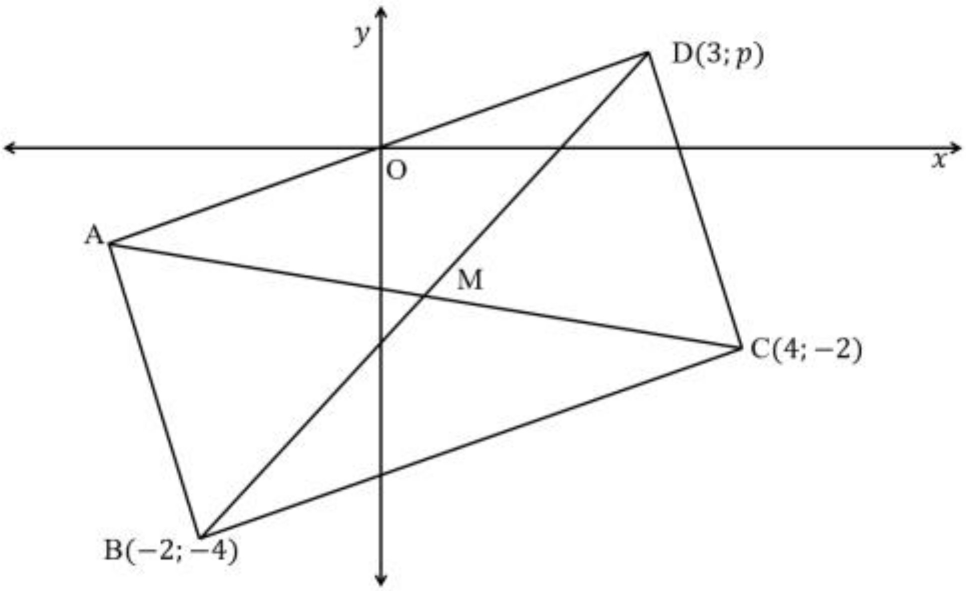
4.4 Prove that KL is parallel to TU. (3)

4.5 Is the point $V\left(-\frac{1}{2}\,;\,7\right)$ inside the circle? Support your answer with calculations. (3)

PAPER G

QUESTION 3

In the diagram A, B$(-2;-4)$, C$(4;-2)$ and D$(3;\ p\ )$ are the vertices of a rectangle. The diagonals AC and BD intersect at M.



3.1 Given that the length of AC is $\sqrt{50}$ units, show that $p = 1$. (4)

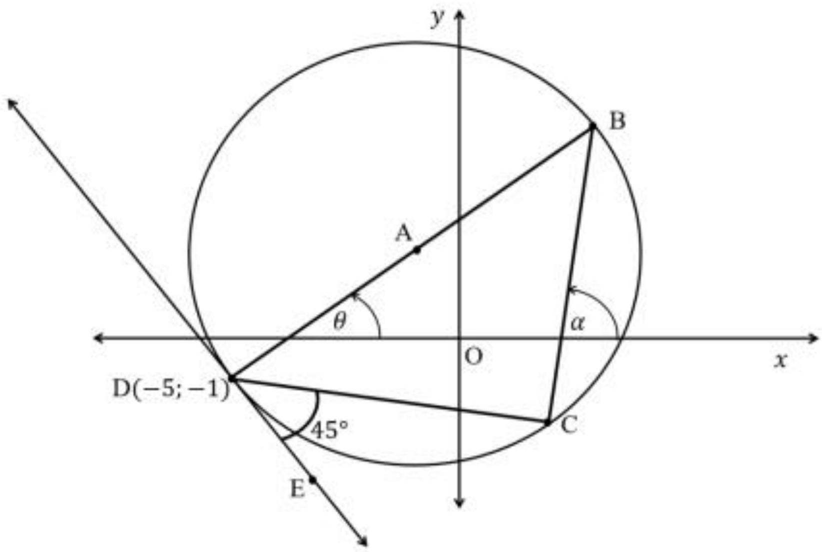
3.2 Determine the coordinates of M. (3)

3.3 Calculate the gradient of DC (2)

3.4 Determine the equation of line AB in the form $y = mx + c$. (2)

QUESTION 4

In the diagram is the circle with equation $(x+1)^2 + (y-2)^2 = 25$.
DB is the diameter of the circle and A the centre of the circle. DE is a tangent to the circle at $D(-5;-1)$. The angle $E\hat{D}C = 45°$. The inclination angles of AD and BC is θ and α respectively. B and C are points on the circumference of the circle.



4.1 Determine:

4.1.1 The coordinates of A, the centre of the circle. (2)

4.1.2 The coordinates of B (3)

4.1.3 The gradient of AD. (2)

4.1.4 The value of θ, the inclination angle of AD. (2)

4.1.5 The equation of the tangent DE. (3)

4.2 Calculate the gradient of BC. (4)

4.3 Another circle with equation $x^2 + y^2 - 6x + 2y = 8$ is given.

Show that:

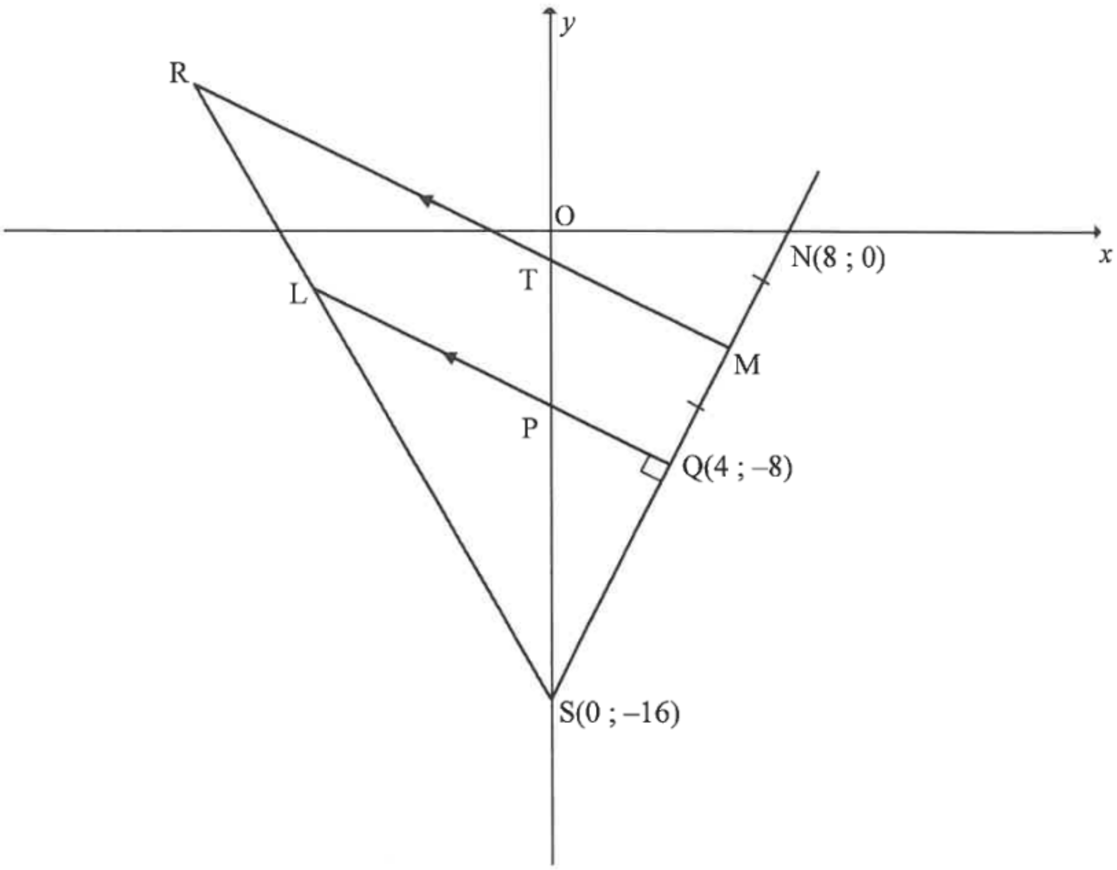
4.3.1 The coordinates of the centre of the circle is $M(3;-1)$. (4)

4.3.2 The two circles will intersect each other. Show all calculations. (4)

PAPER H

QUESTION 3

In the diagram, S(0 ; –16), L and Q(4 ; –8) are the vertices of ΔSLQ having LQ perpendicular to SQ. SL and SQ are produced to points R and M respectively such that RM || LQ. SM produced cuts the *x*-axis at N(8 ; 0). QM = MN. T and P are the *y*-intercepts of RM and LQ respectively.



3.1 Calculate the coordinates of M. (2)

3.2 Calculate the gradient of NS. (2)

3.3 Show that the equation of line LQ is $y = -\frac{1}{2}x - 6$. (3)

3.4 Determine the equation of a circle having centre at O, the origin, and also passing through S. (2)

3.5 Calculate the coordinates of T. (3)

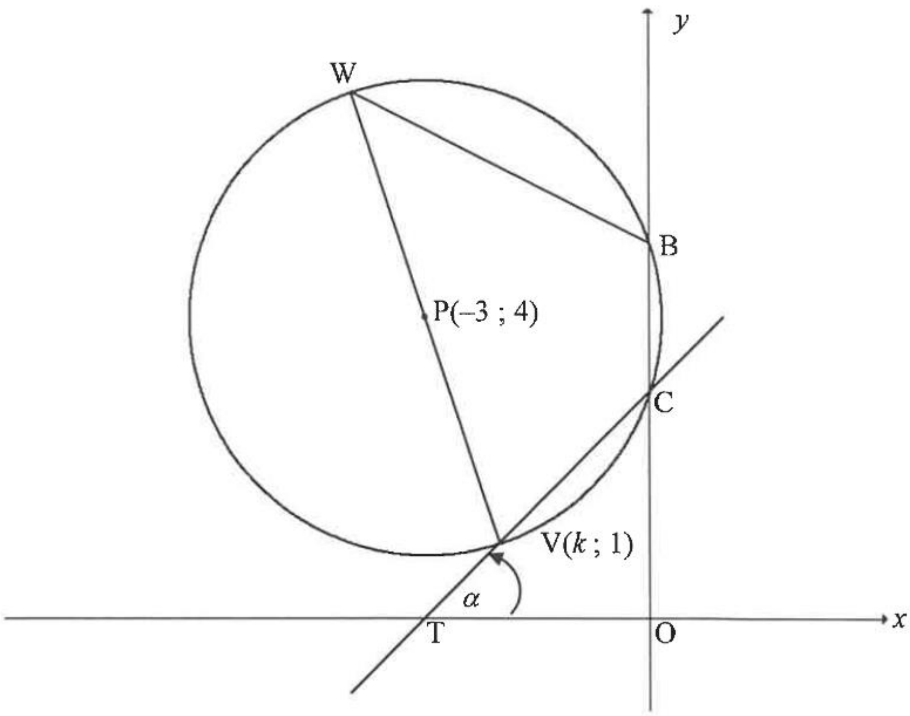
3.6 Determine $\frac{LS}{RS}$. (3)

3.7 Calculate the area of PTMQ. (4)

[19]

QUESTION 4

In the diagram, P(–3 ; 4) is the centre of the circle. V(k ; 1) and W are the endpoints of a diameter. The circle intersects the y-axis at B and C. BCVW is a cyclic quadrilateral. CV is produced to intersect the x-axis at T. $\text{O}\hat{\text{T}}\text{C} = \alpha$.



4.1 The radius of the circle is $\sqrt{10}$. Calculate the value of k if point V is to the right of point P. Clearly show ALL calculations. (5)

4.2 The equation of the circle is given as $x^2+6x+y^2-8y+15=0$. Calculate the length of BC. (4)

4.3 If $k=-2$, calculate the size of:

4.3.1 α (3)

4.3.2 $V\hat{W}B$ (2)

4.4 A new circle is obtained when the given circle is reflected about the line $y=1$.

Determine the:

4.4.1 Coordinates of Q, the centre of the new circle (2)

4.4.2 Equation of the new circle in the form $(x-a)^2+(y-b)^2=r^2$ (2)

4.4.3 Equations of the lines drawn parallel to the y-axis and passing through the points of intersection of the two circles (2)

[20]

PAPER I

Question 2

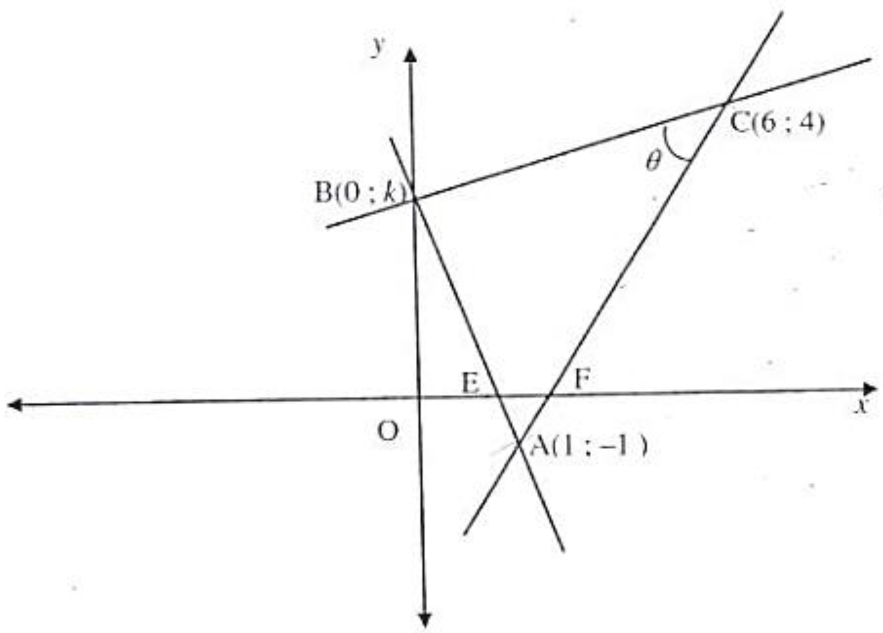
2.2 A circle is represented by the equation $x^2 + 2x + y^2 - 4y - 5 = 0$.

2.2.1 A transformation moves every point 2 units to the left and 4 units up. Determine the equation of the new circle after the transformation. (3)

2.2.2 Does the origin lie within the new circle? Give a reason for your answer. (2)

QUESTION 3

In the diagram, $A(1\,;-1)$, $B(0\,;k)$ and $C(6\,;4)$ are the vertices of ΔABC. The equations of the sides AB and AC are $y+3x-2=0$ and $y=x-2$ respectively. AB cuts the x-axis at E and AC cuts the x-axis at F.



3.1 Write down the value of k. (2)

3.2 Calculate the length of AC and leave your answer in simplest surd form. (2)

3.3 Prove that $\hat{ABC} = 90°$. (3)

3.4 Calculate the size of θ. (5)

3.5 Determine the equation of the circle passing through A, B and C in the form $(x-a)^2+(y-b)^2=r^2$. (4)

3.6 If D is a point in the first quadrant, calculate the coordinates of D such that ABCD in that order, forms a parallelogram. (4)

QUESTION 5

5.1 The equation of a circle is $x^2+y^2-8x+6y=15$.

5.1.1 Prove that the point (2 ; – 9) is on the circumference of the circle. (2)

5.1.2 Determine an equation of the tangent to the circle at the point (2 ; – 9). (7)

5.2 Calculate the length of the tangent AB drawn from the point A(6 ; 4) to the circle with equation $(x-3)^2+(y+1)^2=10$. (5)

QUESTION 6

6.1 Determine the centre and radius of the circle with the equation $x^2+y^2+8x+4y-38=0$. (4)

6.2 A second circle has the equation $(x-4)^2+(y-6)^2=26$. Calculate the distance between the centres of the two circles. (2)

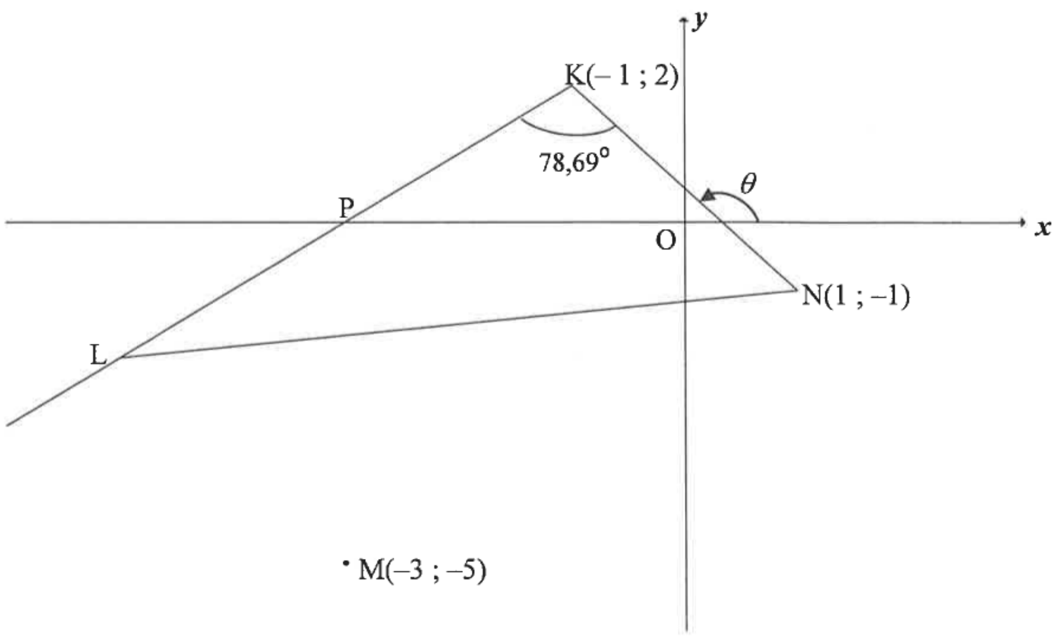
6.3 Hence, show that the circles described in QUESTION 6.1 and QUESTION 6.2 intersect each other. (3)

6.4 Show that the two circles intersect along the line $y = -x+4$. (4)

PAPER J

QUESTION 3

In the diagram, K(– 1 ; 2), L and N(1 ; – 1) are vertices of ΔKLN such that $L\hat{K}N = 78{,}69°$. KL intersects the x-axis at P. KL is produced. The inclination of KN is θ. The coordinates of M are (–3 ; –5).



3.1 Calculate:

3.1.1 The gradient of KN (2)

3.1.2 The size of θ, the inclination of KN (2)

3.2 Show that the gradient of KL is equal to 1. (2)

3.3 Determine the equation of the straight line KL in the form $y = mx + c$. (2)

3.4 Calculate the length of KN. (2)

3.5 It is further given that KN = LM.

3.5.1 Calculate the possible coordinates of L. (5)

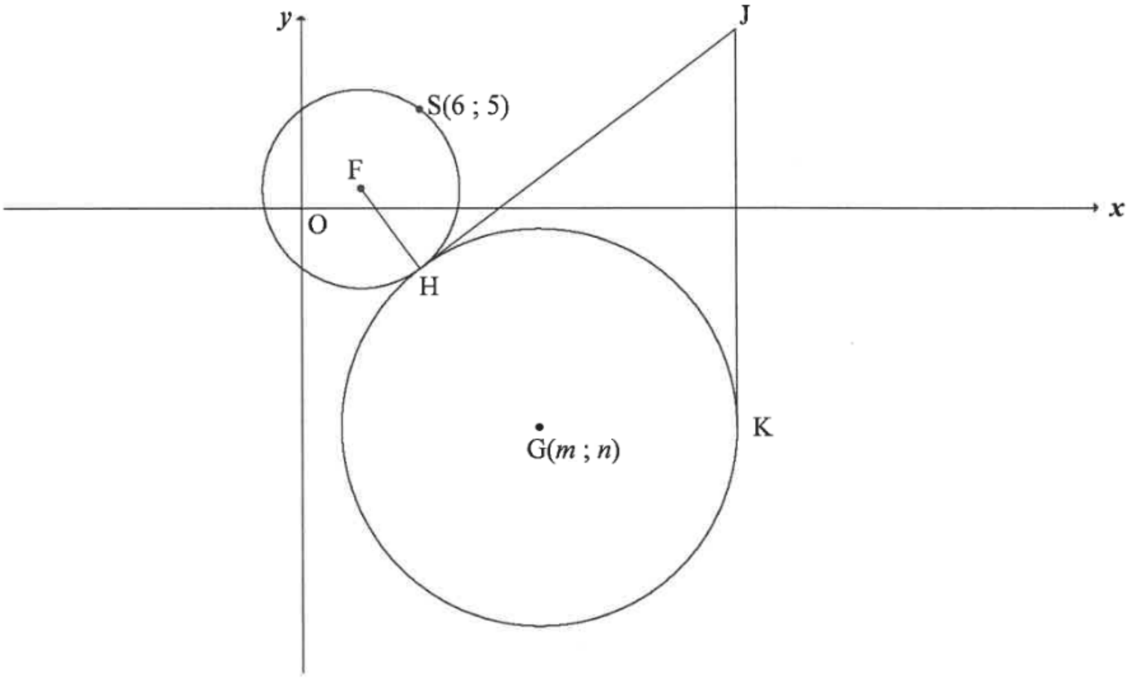
3.5.2 Determine the coordinates of L if it is given that KLMN is a parallelogram. (3)

3.6 T is a point on KL produced. TM is drawn such that TM = LM. Calculate the area of ΔKTN. (4)

[22]

QUESTION 4

In the diagram, the equation of the circle with centre F is $(x-3)^2+(y-1)^2=r^2$. S(6 ; 5) is a point on the circle with centre F. Another circle with centre G(m ; n) in the 4th quadrant touches the circle with centre F, at H such that FH : HG = 1 : 2. The point J lies in the first quadrant such that HJ is a common tangent to both these circles. JK is a tangent to the larger circle at K.



4.1 Write down the coordinates of F. (2)

4.2 Calculate the length of FS. (2)

4.3 Write down the length of HG. (1)

4.4 Give a reason why JH = JK. (1)

4.5 Determine:

4.5.1 The distance FJ, with reasons, if it is given that JK = 20 (4)

4.5.2 The equation of the circle with centre G in terms of m and n in the form $(x-a)^2+(y-b)^2=r^2$ (1)

4.5.3 The coordinates of G, if it is further given that the equation of tangent JK is $x=22$ (7)

[18]

TRIGONOMETRY

1. The reciprocal ratios cosec θ, sec θ and cot θ can be used by candidates in the answering of problems but will not be explicitly tested.
2. The focus of trigonometric graphs is on the relationships, simplification and determining points of intersection by solving equations, although characteristics of the graphs should not be excluded.

Important Formulae from Information Sheet

$$\text{In } \Delta ABC: \ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad a^2 = b^2 + c^2 - 2bc.\cos A \qquad area\ \Delta ABC = \frac{1}{2}ab.\sin C$$

$$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases} \qquad \sin 2\alpha = 2\sin\alpha.\cos\alpha$$

Additional Important Formulae (not on Information Sheet)

In a right-angled triangle:

1. Pythagoras theorem

$(hypotenuse)^2 = (opposite)^2 + (adjacent)^2$

2. Trigonometric ratios

$\sin\alpha = \frac{opposite}{hypotenuse}$ $\qquad$ $\cos\alpha = \frac{adjacent}{hypotenuse}$ $\qquad$ $\tan\alpha = \frac{opposite}{adjacent}$

Other identidities

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$\sin^2\alpha + \cos^2\alpha = 1$ (Pythagorean identity)

Reduction

It is very important to always include the ASTC diagram; clearly showing the signs of each ratio and the general angles in each of the 4 quadrants.

When solving problems:

1. Always solve ratio by ratio rather than looking at the problem as a whole.
2. For each ratio address the following questions:
    i. The angle I am dealing with is in which quadrant
    ii. What is the sign of the ratio in question in the identified quadrant
    iii. Will the ratio change or not (co-ratios)
3. After reductions are done; change everything to $\sin\theta$ and $\cos\theta$ by changing wherever there is $\tan\theta$
4. Check for other relevant identities (double angle, compound angles, Pythagorean identity)
5. Algebraic manipulation to arrive at the answer

Cartesian Plane

Always make sure that the provided ratio is in the form of a fraction; that is; $\sin\alpha = \frac{y}{r}$, $\cos\alpha = \frac{x}{r}$ and $\tan\alpha = \frac{y}{x}$ as this will help to identify the 2 sides of the right-angled triangle.

It is very important to first draw and complete your right-angled triangle (all 3 angles and 3 sides must be done) in the relevant quadrant.

Hints:

i. The terminal is always drawn from the origin.
ii. Perpendicular is always drawn with respect to the horizontal axis (x-axis).
iii. The angle provided in the question is always measured in an anticlockwise direction from the positive x-axis to the terminal.
iv. To solve for 3rd angle; use the sum angles of a triangle.
v. To solve for 3rd side use Pythagoras theorem ($r^2 = x^2 + y^2$)

Take note: r is always positive while x and y change sign depending on in which quadrant is the terminal.

Ratios asked always relate to the angles in the solved right angled triangle but in following ways:

i. Exact angles in the solved right-angled triangle(s)

ii. There could be need for:

- Reduction
- Double angles
- Compound angles related to special angles
- Compound angles related to angles in the solved right-angled triangle

Identities

1. Recommend that always choose the complex side so that it is simplified to be the same as the simpler side.
2. Reduction can be incorporated where necessary.
3. Express everything in terms of $\sin\theta$ and $\cos\theta$ by changing wherever there is $\tan\theta$.
4. Check for other relevant identities (double angle, compound angles, Pythagorean identity)
5. Algebraic manipulation to arrive at the answer.

Trigonometric Equations

Whenever solving any form of trigonometric equation, it is worth noting that it must always reduce to ratio being equated to a constant; that is; $\sin\theta = a$, $\cos\theta = b$ and $\tan\theta = c$.

Forms of trigonometric equations:

1. Simple equations; for example, $\sin\theta = a$, $\cos\theta = b$, $\tan\theta = c$ and $\sin\theta = \cos\theta$
2. Co-ratio equation; for example, $\sin\theta = \cos 2\theta$
3. Quadratic (hidden quadratic equations)

General solution

$\sin\theta = a$	$\cos\theta = b$	$\tan\theta = c$
$\theta = \sin^{-1} a + k.360°$ or	$\theta = \pm\cos^{-1} b + k.360°$	$\theta = \tan^{-1} c + k.180°$
$\theta = 180° - \sin^{-1} a + k.360°$		

Where k in an integer

Particular solution

Substitute integers in the general solution and pick values that lie within the specified range.

3-D Problems

Always remember that you are solving triangles.

Procedure:

1. Extract all the triangles forming the 3-D.
2. Solve the triangles:
    i. Pythagoras theorem
        - Solve third side of a ***right-angled triangle*** when given any 2 sides

    ii. Trigonometric ratios
        - Solve any side of a ***right-angled tringle*** when given at least 1 side and 1 angle (other than the right-angle).
        - Solve any angle of a **right-angled triangle** when given at least 2 sides.

    iii. Sum of angles in a triangle
        - Solve for the 3rd angle of ***any triangle*** when 2 angles are known/ given

    iv. Sin rule
        - Solve for any side of ***any triangle*** when given at least 1 side and 2 angles
        - Solve for angles of **any triangle** when given at least 2 sides and a non-included angle.

    v. Cosine rule
        - Solve for the 3rd side given any 2 sides of ***any triangle*** and an included angle
        - Solve for any angle of ***any triangle*** when given all the 3 sides of a triangle

    vi. Area rule
        - To determine the area of ***any triangle***
        - To determine the included angle of **any triangle** when at least 2 sides and the area are provided

To determine a side of any triangle when a side and an angle are provided. The given side and the calculated side must form the given angle.

PAPER A

QUESTION 5

5.1 Given that $\sqrt{13}\sin x + 3 = 0$, where $x \in (90°\,;\,270°)$.

Without using a calculator, determine the value of:

5.1.1 $\sin(360° + x)$ (2)

5.1.2 $\tan x$ (3)

5.1.3 $\cos(180° + x)$ (2)

5.2 Determine the value of the following expression, **without using a calculator**:

$$\frac{\cos(90° + \theta)}{\sin(\theta - 180°) + 3\sin(-\theta)}$$ (5)

5.3 Determine the general solution of the following equation:

$(\cos x + 2\sin x)(3\sin 2x - 1) = 0$ (6)

5.4 Given the identity: $\cos(x + y).\cos(x - y) = 1 - \sin^2 x - \sin^2 y$

5.4.1 Prove the identity. (4)

5.4.2 Hence, determine the value of $1 - \sin^2 45° - \sin^2 15°$, **without using a calculator**. (3)

5.5 Consider the trigonometric expression: $16\sin x.\cos^3 x - 8\sin x.\cos x$

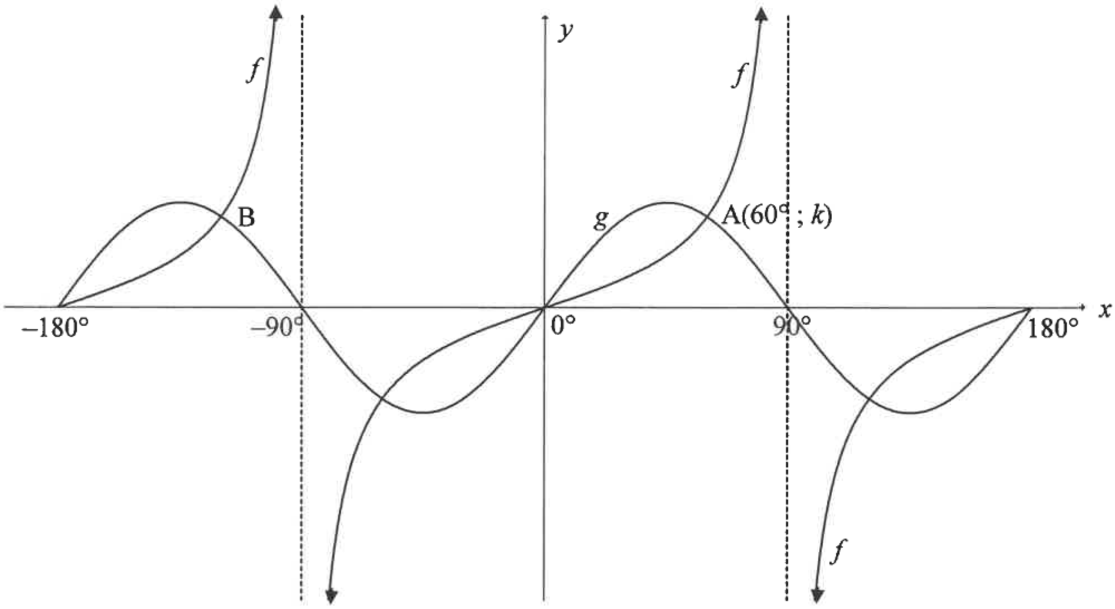
5.5.1 Rewrite the expression as a single trigonometric ratio. (4)

5.5.2 For which value of x in the interval $x \in [0°\,;\,90°]$ will $16\sin x.\cos^3 x - 8\sin x.\cos x$ have its minimum value? (1)

[30]

QUESTION 6

In the diagram below, the graphs of $f(x) = \tan x$ and $g(x) = 2\sin 2x$ are drawn for the interval $x \in [-180° ; 180°]$ A$(60° ; k)$ and B are two points of intersection of f and g.



6.1 Write down the period of g. (1)

6.2 Calculate the:

6.2.1 Value of k (1)

6.2.2 Coordinates of B (1)

6.3 Write down the range of $2g(x)$. (2)

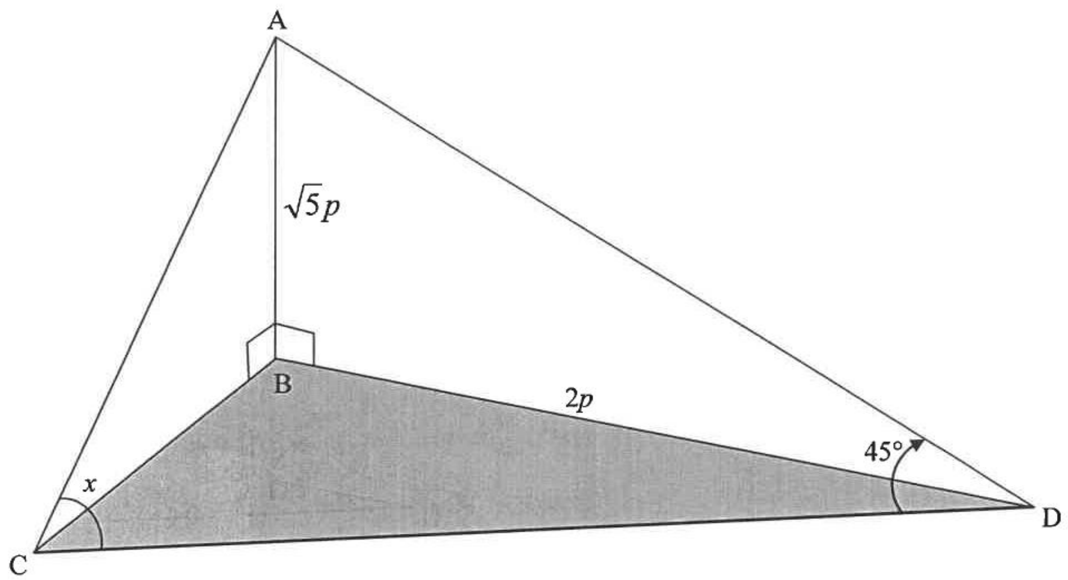
6.4 For which values of x will $g(x+5°) - f(x+5°) \leq 0$ in the interval $x \in [-90° ; 0°]$? (2)

6.5 Determine the values of p for which $\sin x.\cos x = p$ will have exactly two real roots in the interval $x \in [-180° ; 180°]$. (3)

[10]

QUESTION 7

AB is a vertical flagpole that is $\sqrt{5}p$ metres long. AC and AD are two cables anchoring the flagpole. B, C and D are in the same horizontal plane.
$BD = 2p$ metres, $A\hat{C}D = x$ and $A\hat{D}C = 45°$.



7.1 Determine the length of AD in terms of p. (2)

7.2 Show that the length of $CD = \dfrac{3p(\sin x + \cos x)}{\sqrt{2}\sin x}$. (5)

7.3 If it is further given that $p = 10$ and $x = 110°$, calculate the area of ΔADC. (3)
[10]

PAPER B

QUESTION 5

5.1 Simplify the expression to a **single trigonometric term:**

$\tan(-x).\cos x.\sin(x-180°)-1$ (5)

5.2 Given: $\cos 35° = m$

Without using a calculator, determine the value of EACH of the following in terms of m:

5.2.1 $\cos 215°$ (2)

5.2.2 $\sin 20°$ (3)

5.3 Determine the general solution of:

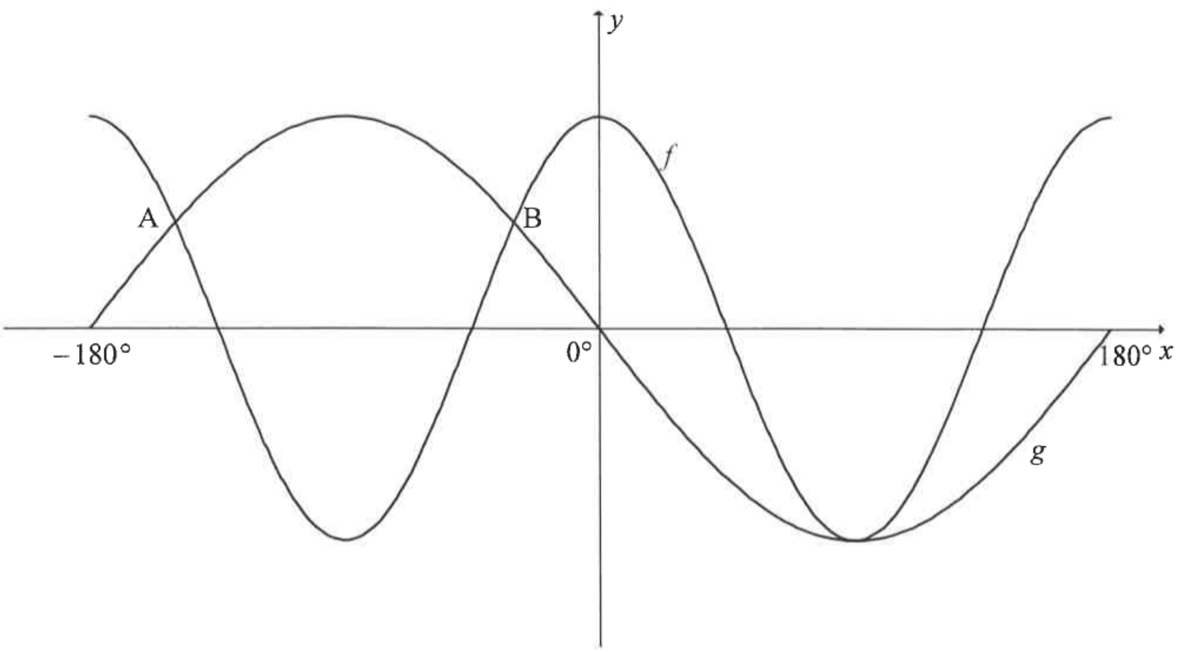
$\cos 4x.\cos x + \sin x.\sin 4x = -0{,}7$ (4)

5.4 Prove the identity: $\dfrac{\sin 4x.\cos 2x - 2\cos 4x.\sin x.\cos x}{\tan 2x} = \cos^2 x - \sin^2 x$ (4)

[18]

QUESTION 6

In the diagram below, the graphs of $f(x) = \cos 2x$ and $g(x) = -\sin x$ are drawn for the interval $x \in [-180° \,;\, 180°]$ A and B are two points of intersection of f and g.



6.1 **Without using a calculator**, determine the values of x for which $\cos 2x = -\sin x$ in the interval $x \in [-180° \; ; \; 180°]$. (6)

6.2 Use the graphs above to answer the following questions:

6.2.1 How many degrees apart are points A and B from each other? (2)

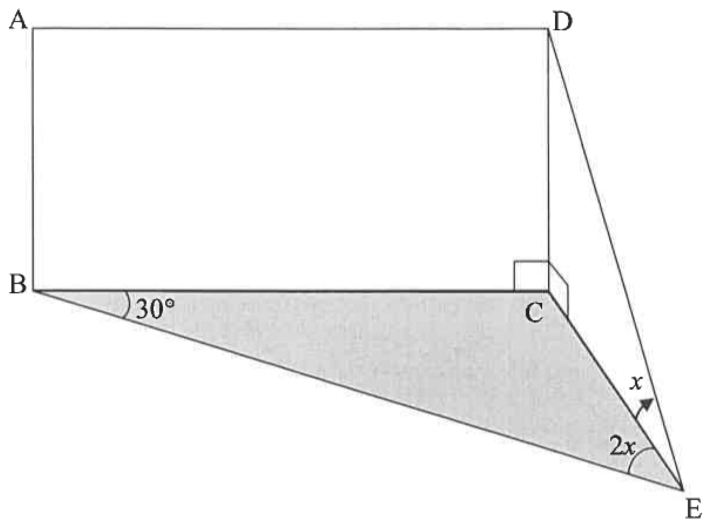
6.2.2 For which values of x in the given interval will $f'(x).g'(x) > 0$? (2)

6.2.3 Determine the values of k for which $\cos 2x + 3 = k$ will have no solution. (3)

[13]

QUESTION 7

Points B, C and E lie in the same horizontal plane. ABCD is a rectangular piece of board. CDE is a triangular piece of board having a right angle at C. Each piece of board is placed perpendicular to the horizontal plane and joined along DC, as shown in the diagram. The angle of elevation from E to D is x. $B\hat{E}C = 2x$ and $E\hat{B}C = 30°$.



7.1 Show that $DC = \dfrac{BC}{4\cos^2 x}$ (6)

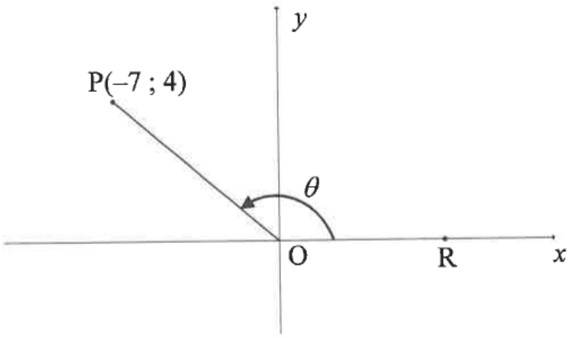
7.2 If $x = 30°$, show that the area of $ABCD = 3AB^2$. (3)

[9]

PAPER C

QUESTION 5

5.1 In the diagram below, P(–7 ; 4) is a point in the Cartesian plane. R is a point on the positive x-axis such that obtuse $\hat{POR} = \theta$.



Calculate, **without using a calculator**, the:

5.1.1 Length OP (2)

5.1.2 Value of:

(a) $\tan\theta$ (1)

(b) $\cos(\theta - 180°)$ (2)

5.2 Determine the general solution of: $\sin x\cos x + \sin x = 3\cos^2 x + 3\cos x$ (7)

5.3 Given the identity: $\dfrac{\sin 3x}{1-\cos 3x} = \dfrac{1+\cos 3x}{\sin 3x}$

5.3.1 Prove the identity given above. (3)

5.3.2 Determine the values of x, in the interval $x \in [0° ; 60°]$, for which the identity will be undefined. (3)

[18]

QUESTION 6

6.1 **Without using a calculator**, simplify the following expression to a single trigonometric term:

$$\frac{\sin 10^\circ}{\cos 440^\circ} + \tan(360^\circ - \theta).\sin 2\theta$$ (6)

6.2 Given: $\sin(60^\circ + 2x) + \sin(60^\circ - 2x)$

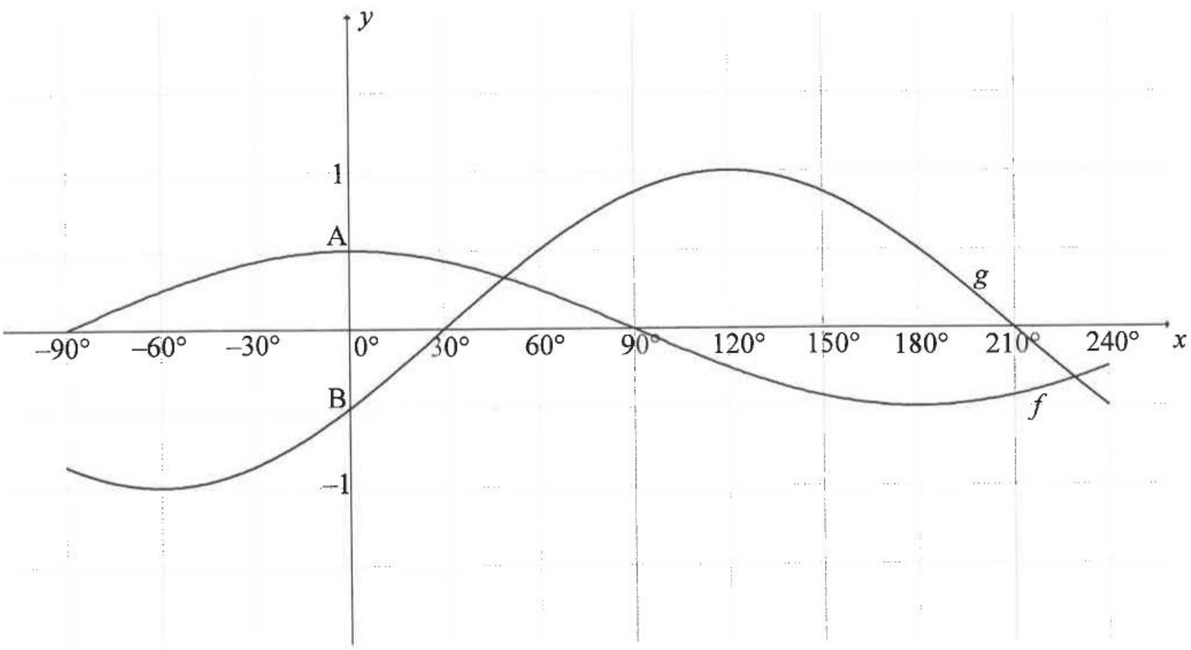
6.2.1 Calculate the value of k if $\sin(60^\circ + 2x) + \sin(60^\circ - 2x) = k\cos 2x$. (3)

6.2.2 If $\cos x = \sqrt{t}$, **without using a calculator**, determine the value of $\tan 60^\circ[\sin(60^\circ + 2x) + \sin(60^\circ - 2x)]$ in terms of t. (3)

[12]

QUESTION 7

In the diagram below, the graphs of $f(x) = \frac{1}{2}\cos x$ and $g(x) = \sin(x - 30^\circ)$ are drawn for the interval $x \in [-90^\circ\,;\,240^\circ]$. A and B are the y-intercepts of f and g respectively.



7.1 Determine the length of AB. (2)

7.2 Write down the range of $3f(x)+2$. (2)

7.3 Read off from the graphs a value of x for which $g(x)-f(x)=\frac{\sqrt{3}}{2}$. (2)

7.4 For which values of x, in the interval $x\in[-90°\,;\,240°]$, will:

7.4.1 $f(x).g(x)>0$ (2)

7.4.2 $g'(x-5°)>0$ (2)
[10]

QUESTION 8

FIGURE I shows a ramp leading to the entrance of a building. B, C and D lie on the same horizontal plane. The perpendicular height (AC) of the ramp is 0,5 m and the angle of elevation from B to A is 15°. The entrance of the building (AE) is 0,915 m wide.

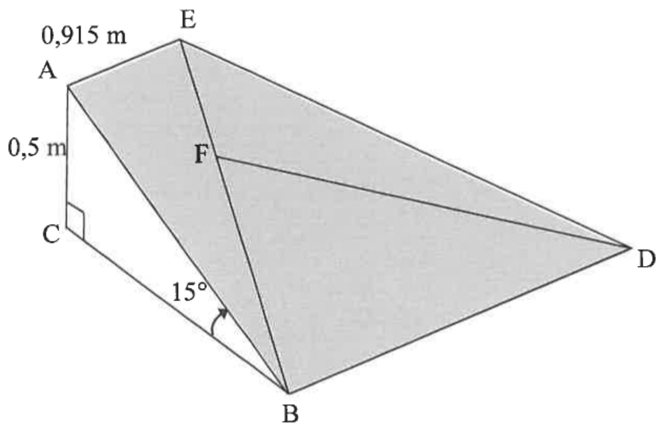


FIGURE I

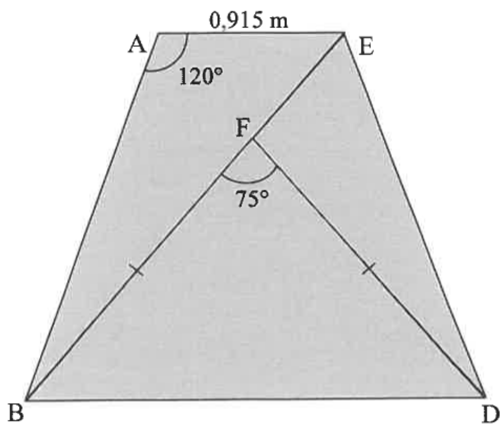


FIGURE II (top view)

8.1 Calculate the length of AB. (2)

8.2 Figure II shows the top view of the ramp. The area of the top of the ramp is divided into three triangles, as shown in the diagram.

If $B\hat{A}E=120°$, calculate the length of BE. (3)

8.3 Calculate the area of ΔBFD if $B\hat{F}D=75°$, $BF=FD$ and $BF=\frac{5}{7}BE$. (3)
[8]

PAPER D

QUESTION 5

5.1 **Without using a calculator**, simplify the following expression to a single trigonometry ratio:

$$\frac{1-\sin(-\theta)\cos(90°+\theta)}{\cos(\theta-360°)}$$ (5)

5.2 Given that $\cos 20° = p$

Without using a calculator, write EACH of the following in terms p:

5.2.1 $\cos 200°$ (2)

5.2.2 $\sin(-70°)$ (2)

5.2.3 $\sin 10°$ (3)

5.3 Determine, **without using a calculator**, the value of:

$$\cos(A+55°)\cos(A+10°)+\sin(A+55°)\sin(A+10°)$$ (3)

5.4 Consider: $\dfrac{\cos 2x+\sin 2x-\cos^2 x}{\sin x-2\cos x}=-\sin x$

5.4.1 Prove the above identity. (3)

5.4.2 Determine the value of $\dfrac{\cos 2x+\sin 2x-\cos^2 x}{-3\sin^2 x+6\sin x\cos x}$ (3)

5.5 Given: $3\tan 4x=-2\cos 4x$

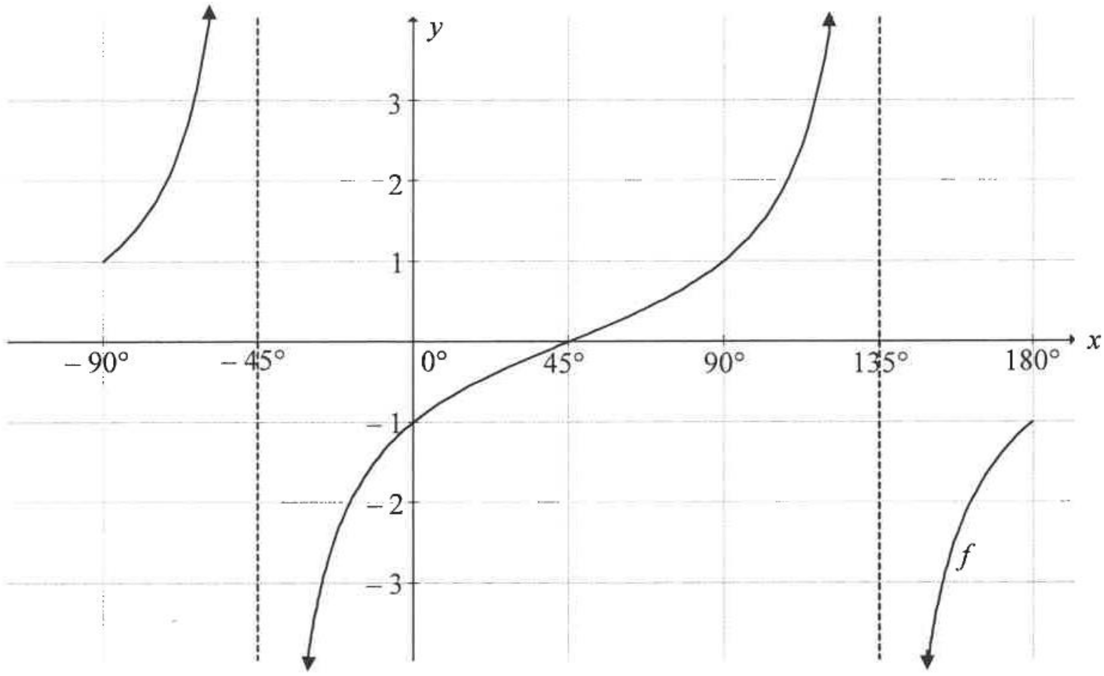
5.5.1 **Without using a calculator**, show that $\sin 4x=-0{,}5$ is the only solution to the above equation. (4)

5.5.2 Hence, determine the general solution of x in the equation $3\tan 4x=-2\cos 4x$ (3)

[28]

QUESTION 6

In the diagram below, the graph of $f(x) = \tan(x - 45°)$ is drawn for $x \in [-90° \,;\, 180°]$.



6.1 Write down the period of f. (1)

6.2 Draw the graph of $g(x) = -\cos 2x$ for the interval $x \in [-90° \,;\, 180°]$ on the grid given in the ANSWER BOOK. Show ALL intercepts with the axes, as well as the minimum and maximum points of the graph. (3)

6.3 Write down the range of g. (1)

6.4 The graph of g is shifted $45°$ to the left to form the graph of h. Determine the equation of h in its simplest form. (2)

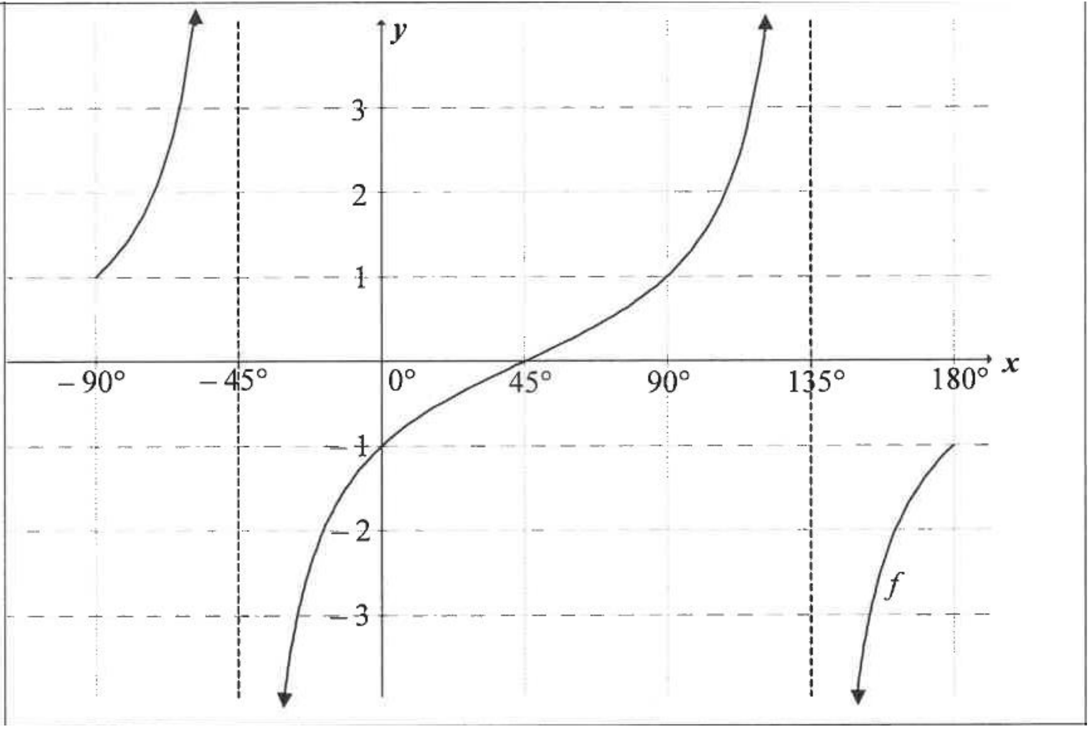
6.5 Use the graph(s) to determine the values of x in the interval $x \in [-90° \,;\, 90°]$ for which:

6.5.1 $f(x) > 1$ (2)

6.5.2 $2\cos 2x - 1 > 0$ (4)

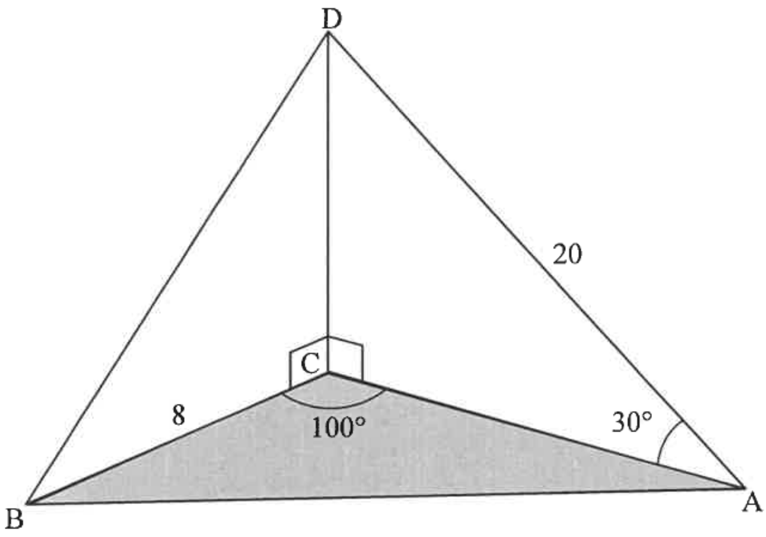
[13]

GRID

6.2	

QUESTION 7

In the diagram, A, B and C are points in the same horizontal plane. D is a point directly above C, that is $DC \perp AC$ and $DC \perp BC$. It is given that $A\hat{C}B = 100°$, $C\hat{A}D = 30°$, $AD = 20$ units and $BC = 8$ units.



7.1 Calculate the length of:

7.1.1 AC (2)

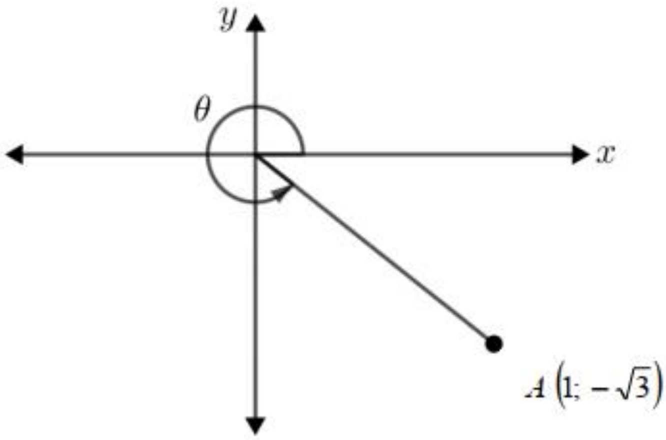
7.1.2 AB (3)

7.2 If it is further given that $A\hat{B}D = 73{,}4°$, calculate the size of $A\hat{D}B$. (3)
[8]

PAPER E

QUESTION 5

5.1 Use the diagram below to calculate, **without the use of a calculator,** the following



5.1.1 $\tan\theta$ (1)

5.1.2 $\sin(-\theta)$ (3)

5.1.3 $\sin(\theta - 60°)$ (4)

5.2 Determine the value of the following trigonometric expression:

$$\frac{\tan(180° - \theta)\sin(90° + \theta)}{\cos 300° \sin(\theta - 360°)}$$ (6)

5.3 Consider: $\dfrac{\cos 2x - 1}{\sin 2x} = -\tan x$

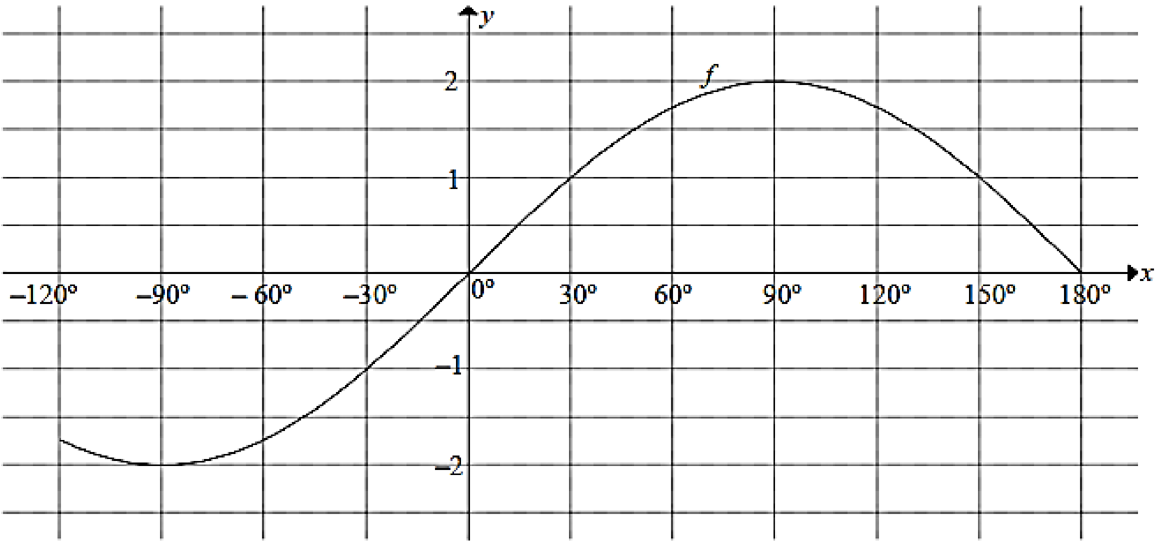
5.3.1 Prove the identity (3)

5.3.2 For which value(s) of x, $0° < x < 360°$, is this identity undefined? (3)

5.3.3 Hence or otherwise, find the general solution of $\dfrac{\sin 4x}{\cos 4x - 1} = 4$. (4)

QUESTION 6

In the diagram below, the graph of $f(x) = 2\sin x$ is drawn for the interval $x \in [-120° ; 180°]$.



6.1 Draw on the same system of axes the graph of $g(x) = \cos(x + 30°)$, for the interval $x \in [-120° ; 180°]$. Show all intercepts with the axes as well as the turning and end Points of the graph. (4)

6.2 Write down the period of *f*. (1)

6.3 For which values of *x* in the interval $x \in [-120° ; 180°]$ is:

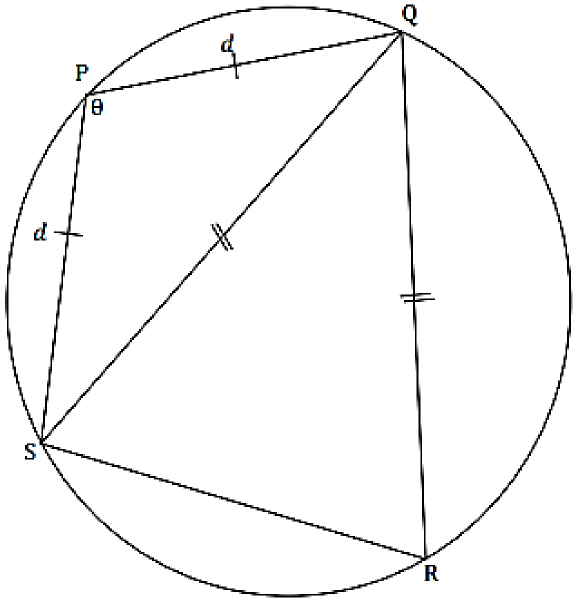
6.3.1 The graph of g decreasing? (2)

6.3.2 $f(x).\ g(x) > 0$? (2)

6.4 If the graph of *g* is moved 60° to the left, determine the equation of the new graph which is formed, in its simplest form. (2)

QUESTION 7

In the diagram, PQRS is a cyclic quadrilateral with QS = QR and PQ = PS = d units. $Q\hat{P}S = \theta$.



Use the diagram to prove that:

7.1. $QS = d\sqrt{2(1-\cos\theta)}$ (2)

7.2 The area of $\Delta QRS = -d^2 \sin 2\theta\,(1-\cos\theta)$ (3)

PAPER F

QUESTION 5

5.1 Calculate the value of $1-4\sin^2 15°$ without the use of a calculator. (5)

5.2 Simplify without the use of a calculator:

$$\frac{\sqrt{3}\sin x.\sin^2 72° + \sin^2 198°.\sqrt{3}\cos(x-90°)}{\tan 120°.\,\sin x}$$ (6)

5.3 Determine the general solution of the following:

$$6\sin x.\cos x + 3\cos x - 4\sin^2 x - 2\sin x = 0 \qquad (7)$$

5.4 Prove that:

$$(1 - \tan A)\left(\frac{\cos A}{\cos 2A}\right) = \frac{1}{\cos A + \sin A} \qquad (4)$$

5.5 If $\sin 2\theta = k$ and $0° < 2\theta < 90°$, determine in terms of k:

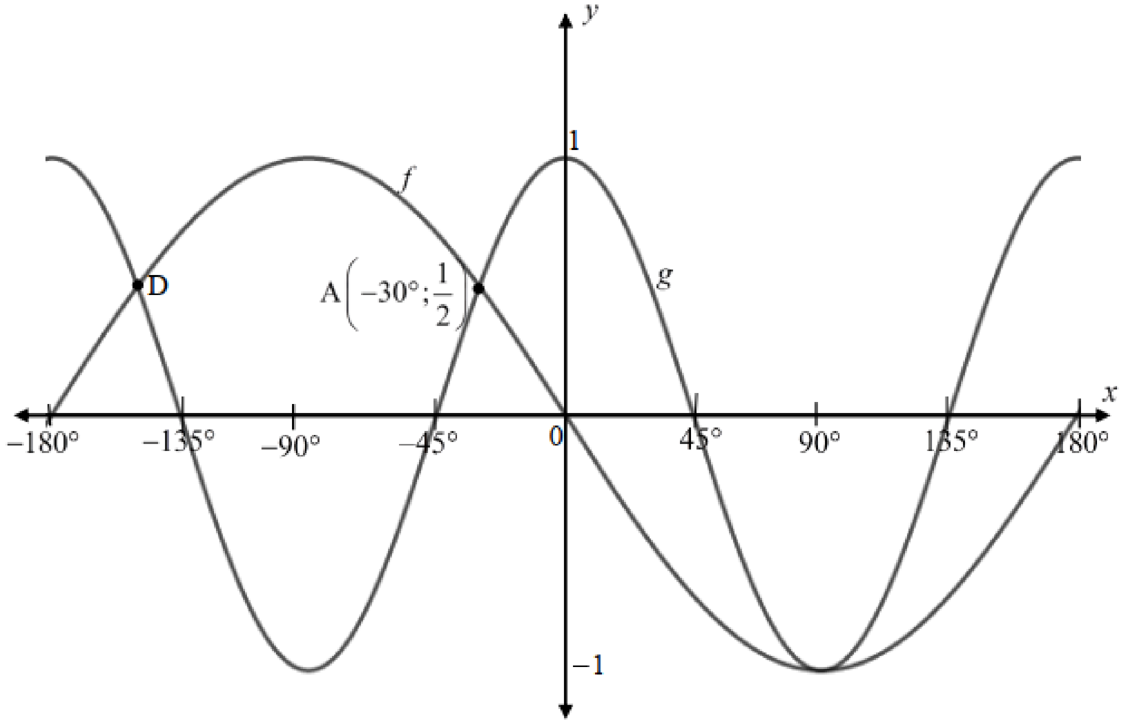
5.5.1 $\cos 2\theta$ (2)

5.5.2 $\dfrac{\sin 2\theta}{\tan \theta}$ (5)

QUESTION 6

The sketch below shows the graphs of $f(x) = a\sin x$ and $g(x) = \cos dx$ for $x \in [-180° \ ; 180°]$.

$A\left(-30° \ ; \frac{1}{2}\right)$ is a point of intersection of f and g.



6.1 Write down the values of a and d. (2)

6.2 Determine the coordinates of D. (1)

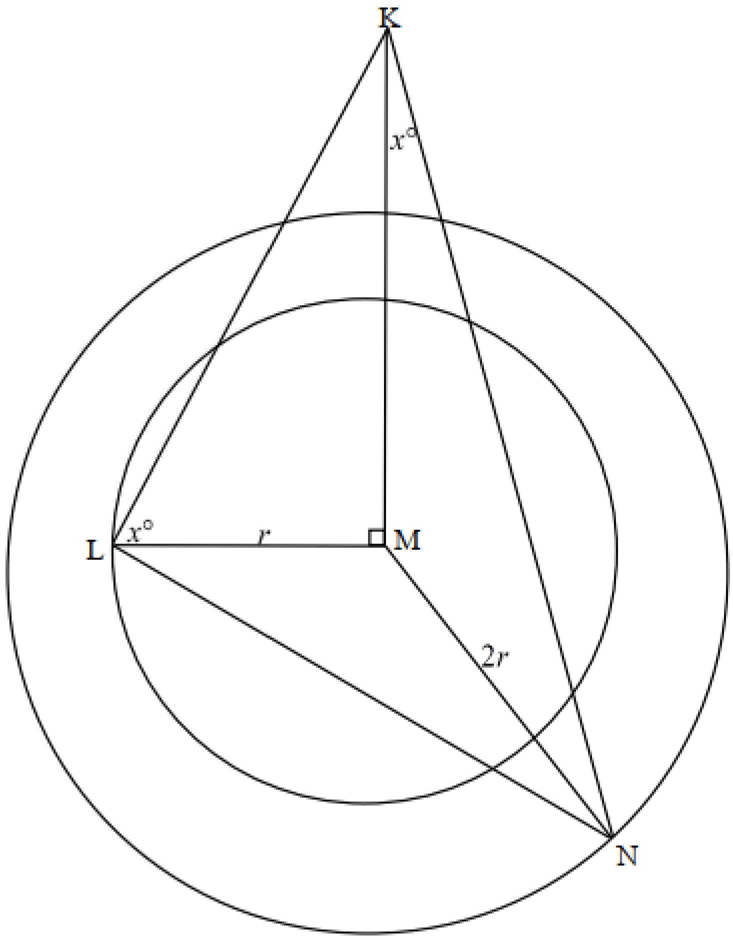
6.3 For which value(s) of x is:

6.3.1 f decreasing for $x \in [-180° ; 180°]$? (2)

6.3.2 $f(x) \cdot g(x) < 0$ for $x \in [-180° ; 0°]$? (2)

QUESTION 7

In the figure below, KM is a vertical flag post set in the centre of two circles which lie on the same horizontal plane. $M\hat{K}N = M\hat{L}K = x°$. The radius of the inner circle $ML = r$ units and the radius of the outer circle $MN = 2r$ units.



7.1 Calculate the value of x. (6)

7.2 If $r = 5m$ and $L\hat{M}N = 110°$, calculate the length of LN. (2)

PAPER G

QUESTION 5

5.1 If $\sin 16° = \frac{1}{\sqrt{1+k^2}}$, express the following in terms of k, **without the use of a calculator.**

5.1.1 $\tan 16°$ (2)

5.1.2 $\cos 32°$ (3)

5.2 Simplify the following expression.

$$\frac{\cos(90°+x)\sin(x-180°)-\cos^2(180°-x)}{\cos(-2x)}$$ (6)

5.3 Calculate the value of the following, **without the use of a calculator.**

$\cos 75°.\cos 45° - \cos 15°.\cos 45°$ (4)

5.4 Given: $\tan\theta\left(\sin 2\theta + \frac{3\cos^2\theta}{\sin\theta}\right) = -2\cos^2\theta + 3\cos\theta + 2$

5.4.1 Prove the identity. (3)

5.4.2 Determine the general solution of:

$$\tan\theta\left(\sin 2\theta + \frac{3\cos^2\theta}{\sin\theta}\right) = 0$$ (4)

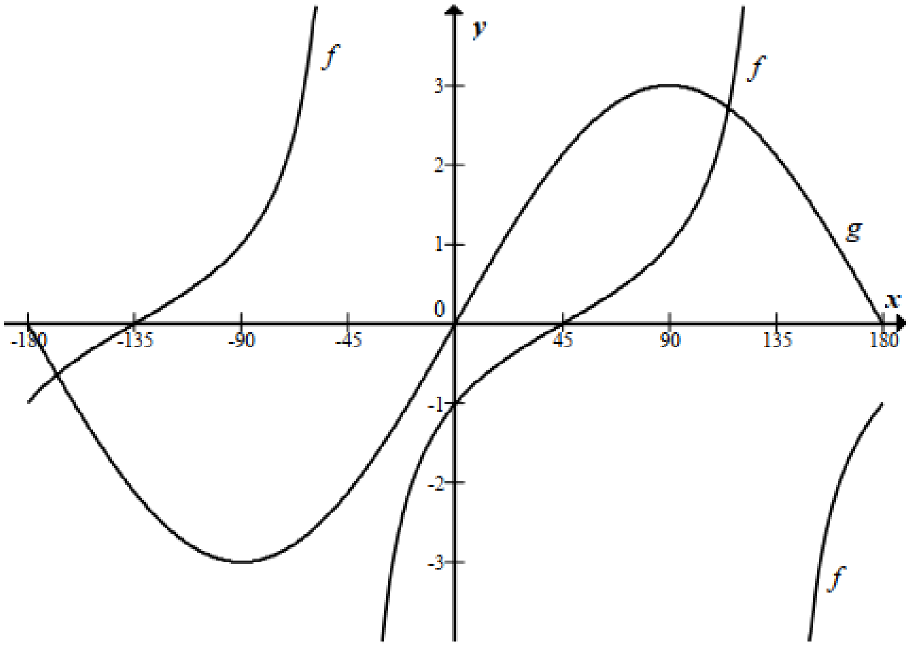
5.5 Solve for a and b:

$\cos(a+b) = -\frac{\sqrt{2}}{2}$ if $a+b \in [0°\,;180°]$

$\cos(a-2b) = \frac{1}{2}$ if $a-2b \in [0°\,;180°]$ (4)

QUESTION 6

Sketched below are the graphs of the functions $f(x) = \tan(x - 45°)$ and $g(x) = 3\sin x$ for $x \in [-180° ; 180°]$.



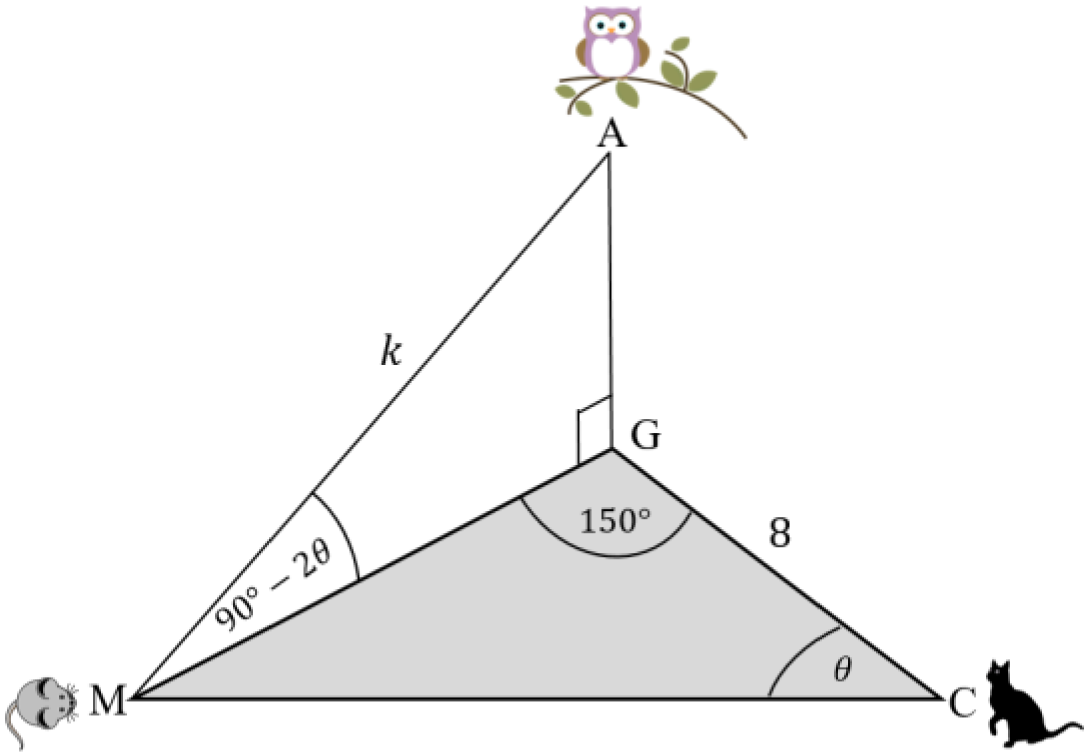
6.1 Write down the equations of the asymptotes of $y = f(x)$ for $x \in [-90° ; 180°]$. (2)

6.2 Describe the transformation of the graph of f to h if $h(x) = \tan(45° - x)$. (2)

6.3 The period of g is reduced to 180° and the amplitude and y-intercept remain the same. Write down the equation of the resulting function. (2)

QUESTION 7

A mouse on the ground is looking up to an owl in a tree and a cat to his right on the ground. The angle of elevation from the mouse to the owl is $(90° - 2\theta)$.
AM $= k$ *units*, GC$=$ 8 *units* , $M\hat{G}C = 150°$ and $M\hat{C}G = \theta$



7.1 Give the size of $M\hat{A}G$ in terms of θ. (1)

7.2 Show that MG$= k \sin 2\theta$ (2)

7.3 Show that MC$= k \cos \theta$ (4)

7.4 Show that the area of ΔMGC$= 2k \sin 2\theta$ (2)

PAPER H

QUESTION 5

5.1 If $\sin 40°.\cos 22° + \cos 40°.\ \sin 22° = k$, determine without the use of a calculator, the value of the following in terms of k.

5.1.1 $\sin 62°$ (2)

5.1.2 $\tan 118°$ (4)

5.1.3 $\sin 14°.\ \cos 14°$ (3)

5.2 Prove the following identity:

$$\frac{1-\cos 2\theta}{\sin 2\theta \times \tan\theta} = 1$$ (4)

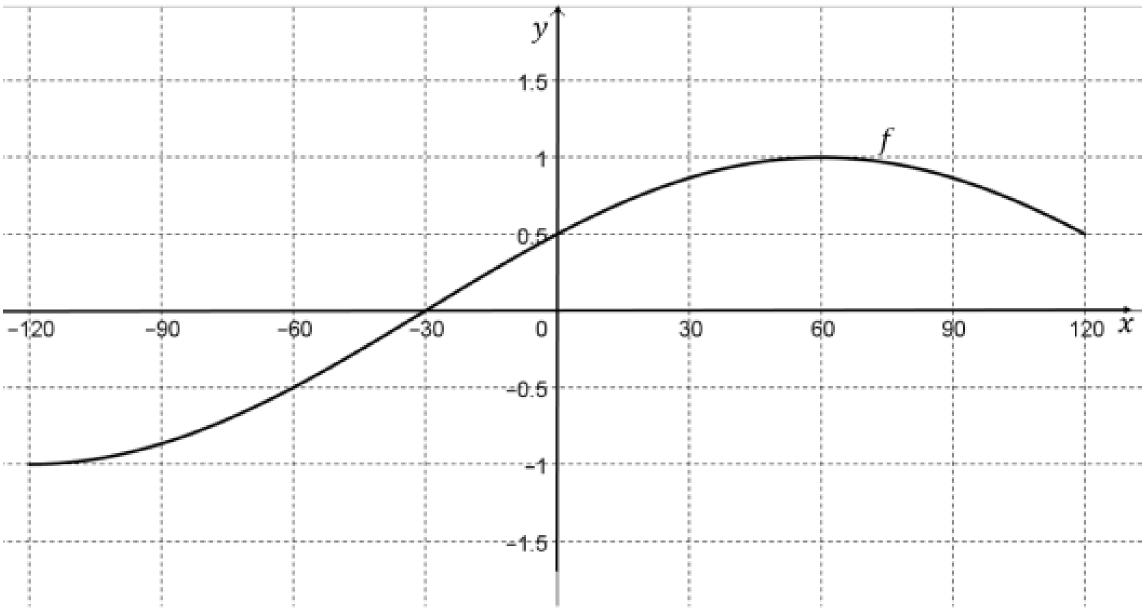
5.3 For which value(s) of A will the following expression be real?

$$\sqrt{\sin(180° + \mathrm{A}).\cos(90° + \mathrm{A}) - \tan 45°}$$

(6)

QUESTION 6

In the diagram is the graph of $f(x) = \sin(x + a)$ for the interval [-120°;120°]



6.1 Determine the numerical value of a. (1)

6.2 On the grid provided in the ANSWER BOOK, draw the graph of $g(x) = \cos(3x)$ for the interval $x \in [-120°; 120°]$. Clearly show ALL intercepts with the axes, the turning point(s) and endpoint(s) of the graph. (4)

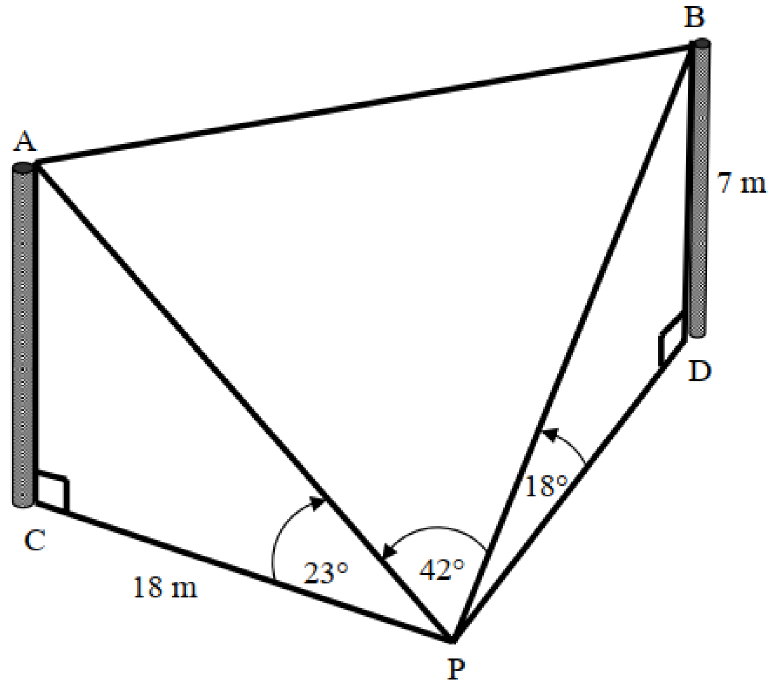
6.3 Determine the general solution for the following: $f(x) = g(x)$ (5)

6.4 Determine the values of x in the interval $x \in [0°; 120°]$, for which $f(x) > g(x)$. (2)

6.5 Describe the transformation from graph g to the graph of $k(x) = \cos(60° - 3x)$. (2)

QUESTION 7

Thandi is standing at point P on the horizontal ground and observes two poles, AC and BD, of different heights. P, C and D are in the same horizontal plane. From P the angles of inclination to the top of the poles A and B are 23° and 18° respectively. Thandi is 18 m from the base of pole AC. The height of pole BD is 7 m.



Calculate, correct to TWO decimal places:

7.1 The distance from Thandi to the top of pole BD (2)

7.2 The distance from Thandi to the top of pole AC (2)

7.3 The distance between the tops of the poles, that is the length of AB, if $A\hat{P}B = 42°$ (4)

QUESTION 8

Consider the functions defined by $f(x) = \sin 2x$ and $g(x) = \frac{1}{2}\tan x$ for $x \in [-90° ; 180°]$.

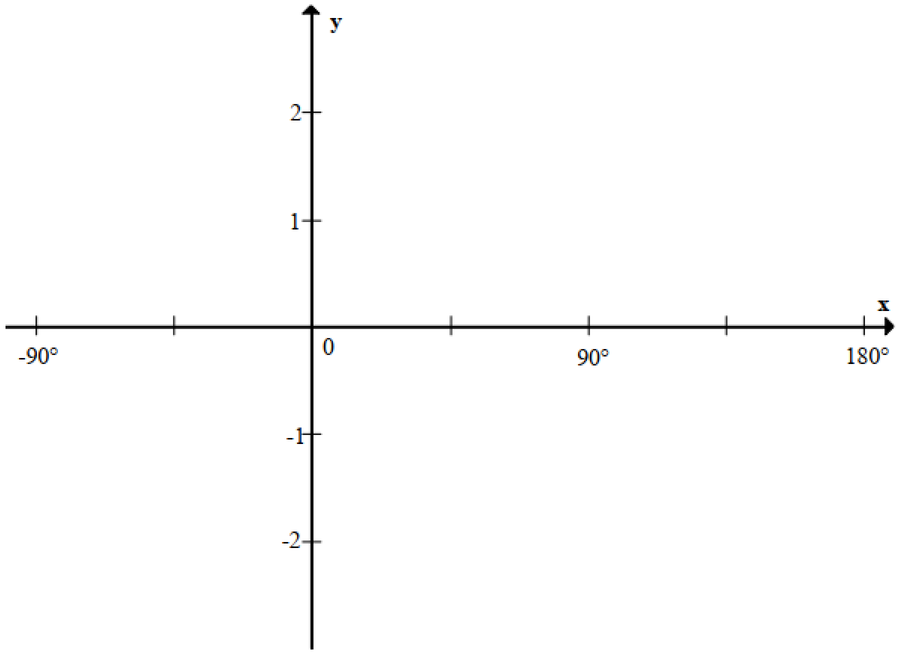
8.1 Sketch the graphs of f and g on the same system of axes on DIAGRAM SHEET 1. (6)

8.2 Calculate the x-coordinates of the points of intersection of f and g. (10)

8.3 Determine the values of x for which $g(x) > f(x)$. (3)

DIAGRAM SHEET 1

QUESTION 8.1



PAPER I

QUESTION 6

6.1 If $\sin 23° = p$, write down the following in terms of p. Do NOT use a calculator.

6.1.1 $\cos 113°$ (2)

6.1.2 $\cos 23°$ (2)

6.1.3 $\sin 46°$ (2)

6. 2 It is known that $13\sin\alpha - 5 = 0$ and $\tan\beta = -\frac{3}{4}$ where $\alpha \in [90°\,;270°]$ and $\beta \in [90°\,;270°]$. Determine, without using a calculator, the values of the following:

6.2.1 $\cos \alpha$ (3)

6.2.2 $\cos(\alpha + \beta)$ (5)

6.3 Solve for $x \in [0°\,;360°]$ if $\frac{1}{2}\cos x = 0{,}435$. (3)

QUESTION 9

9.1 If $4\tan\theta = 3$ and $180° < \theta < 360°$, determine with the aid of a diagram:

9.1.1 $\sin\theta + \cos\theta$ (4)

9.1.2 $\tan 2\theta$ (5)

9.2 9.2.1 Show that: $\dfrac{\cos(360° - x)\tan^2 x}{\sin(x - 180°)\cos(90° + x)} = \dfrac{1}{\cos x}$ (5)

9.2.2 Hence, calculate without the use of a calculator, the value of:

$\dfrac{\cos 330° \tan^2 30°}{\sin(-150°)\cos 120°}$ (Leave your answer in surd form.) (2)

QUESTION 11

Given: $f(x) = 1 + \sin x$ and $g(x) = \cos 2x$

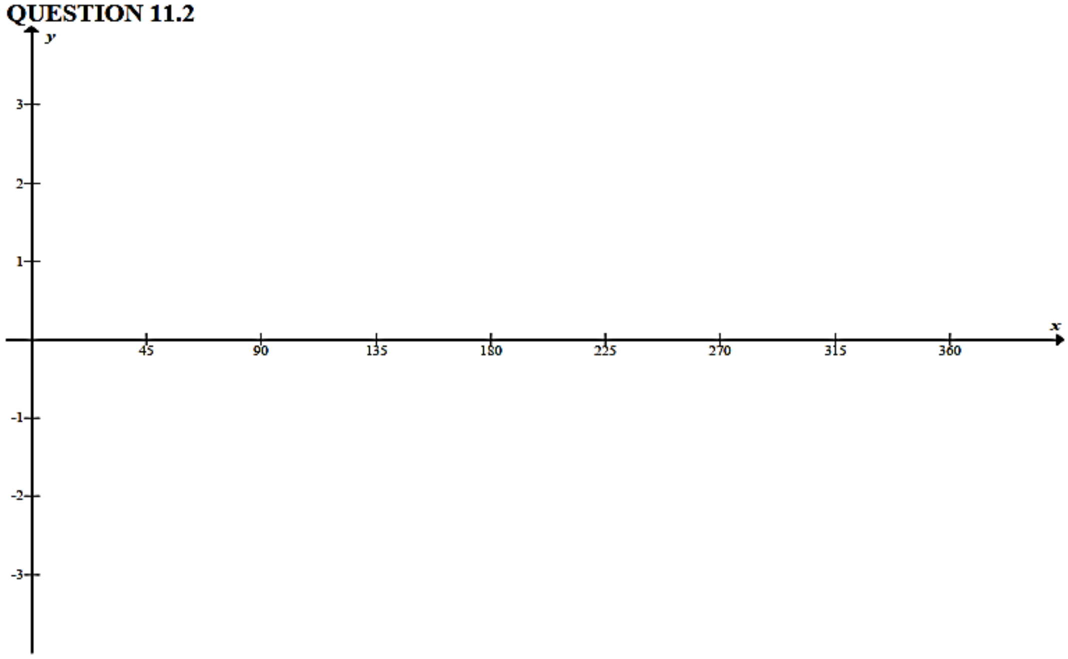
11.1 Calculate the points of intersection of the graphs f and g for $x \in [180°\,;360°]$. (7)

11.2 Draw sketch graphs of f and g for $x \in [180°\,;360°]$ on the same system of axes provided on DIAGRAM SHEET 3. (4)

11.3 For which values of x will $f(x) \leq g(x)$ for $x \in [180°\,;360°]$? (3)

DIAGRAM SHEET 3

QUESTION 11.2



PAPER J

QUESTION 10

10.1 If $\sin 28° = a$ and $\cos 32° = b$, determine the following in terms of a and/or b:

10.1.1 $\cos 28°$ (2)

10.1.2 $\cos 64°$ (3)

10.1.3 $\sin 4°$ (4)

10.2 Prove without the use of a calculator, that if $\sin 28° = a$ and $\cos 32° = b$, then $b\sqrt{1-a^2} - a\sqrt{1-b^2} = \frac{1}{2}$. (4)

QUESTION 11

11.1 If $\sin 61° = \sqrt{p}$, determine the following in terms of p:

11.1.1 $\sin 241°$ (2)

11.1.2 $\cos 61°$ (2)

11.1.3 $\cos 122°$ (3)

11.1.4 $\cos 73° \cos 15° + \sin 73° \sin 15°$ (3)

11.2 11.2.1 Prove the identity:

$$\frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x$$ (6)

11.2.2 Determine a value of x in the interval [0° ; 180°] for which the identity is not valid. (2)

11.3 11.3.1 Given: $\sin x = \cos 2x - 1$. Show that $2\sin^2 x + \sin x = 0$. (1)

11.3.2 Determine the general solution of the equation: $\sin x = \cos 2x - 1$. (6)

11.4 Determine the value of:

$$\tan 1° \times \tan 2° \times \tan 3° \times \tan 4° \times ... \times \tan 87° \times \tan 88° \times \tan 89°.$$ (4)

EUCLID'S GEOMETRY

1. The following proofs of theorems are examinable:
   - The line drawn from the centre of a circle perpendicular to a chord bisects the chord.
   - The line drawn from the centre of a circle that bisects a chord is perpendicular to the chord.
   - The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).
   - The opposite angles of a cyclic quadrilateral are supplementary.
   - The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.
   - A line drawn parallel to one side of a triangle divides the other two sides proportionally.
   - Equiangular triangles are similar.
2. Corollaries derived from the theorems and axioms are necessary in solving riders:
   - Angles in a semi-circle.
   - Equal chords subtend equal angles at the circumference.
   - Equal chords subtend equal angles at the centre.
   - In equal circles, equal chords subtend equal angles at the circumference.
   - In equal circles, equal chords subtend equal angles at the centre.
   - The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle of the quadrilateral.
   - If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.
   - Tangents drawn from a common point outside the circle are equal in length.

3. The theory of quadrilaterals will be integrated into questions in the examination.

Acceptable Reasons

In order to have some kind of uniformity, the use of the following shortened versions of the theorem statements is encouraged.

Theorem statement	**Acceptable reason(s)**
Lines	
The adjacent angles on a straight line are supplementary.	angles on a straight line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adjacent angles supplement
The adjacent angles in a revolution add up to 360°.	angles round a point OR angles in a revolution
Vertically opposite angles are equal.	vertically opposite angles are equal
If AB || CD, then the alternate angles are equal.	alternate angles; AB || CD
If AB || CD, then the corresponding angles are equal.	corresponding angles; AB || CD
If AB || CD, then the co-interior angles are supplementary.	co-interior angles; AB || CD
If the alternate angles between two lines are equal, then the lines are parallel.	alternate angles are equal
If the corresponding angles between two lines are equal, then the lines are parallel.	corresponding angles are equal
If the co-interior angles between two lines are supplementary, then the lines are parallel.	co-interior angles supplement
Triangles	
The interior angles of a triangle are supplementary.	angle sum in a triangle OR sum of angles in a triangle OR Int angles of a triangle
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	exterior angle of a triangle
The angles opposite the equal sides in an isosceles triangle are equal.	angles opposite equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opposite equal angles

In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras OR Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides, then the triangle is right-angled.	Converse Pythagoras OR Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	AAS OR ASA OR SAA
If in two right-angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	RHS
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpoint theorem
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpoint parallel to 2^{nd} side
A line drawn parallel to one side of a triangle divides the other two sides proportionally	line parallel one side of a triangle OR proportionality theorem; name parallel lines
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of a triangle in proportion
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	Similar triangles OR equiangular triangles
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of triangle in proportion

If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	same base; same height OR equal bases; equal height
Circles	
The tangent to a circle is perpendicular to the radius/ diameter of the circle at the point of contact.	Tangent perpendicular to radius OR Tangent perpendicular to diameter
If a line is drawn perpendicular to a radius/ diameter at the point where the radius/ diameter meets the circle, then the line is a tangent to the circle	line perpendicular to radius OR converse Tangent perpendicular to radius OR converse Tangent perpendicular to diameter
The line drawn from the centre of a circle to the midpoint of a chord/ bisecting a chord is perpendicular to the chord.	Line from centre to midpoint of chord OR Line from centre to bisecting a chord
The line drawn from the centre of a circle perpendicular to a chord bisects the chord/ passes through the midpoint of the chord.	line from centre perpendicular to chord
The perpendicular bisector of a chord passes through the centre of the circle.	perpendicular bisector of a chord
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)	angle at centre equal to 2 times angle at circumference OR angle at circumference equal to half angle at centre
The angle subtended by the diameter at the circumference of the circle is 90°.	angles in semi-circle OR diameter subtends right angle OR angle at centre equal to 2 times angle at circumference OR angle at circumference equal to half angle at centre
If the angle subtended by a chord at the circumference of the circle is 90°, then the chord is a diameter.	chord subtends 90° OR converse angles in semi-circle
Angles subtended by a chord of the circle, on the same side of the chord, are equal	angles in the same segment

If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	Line subtends equal angles OR converse angles in the same segment
Equal chords subtend equal angles at the circumference of the circle.	equal chords; equal angles
Equal chords subtend equal angles at the centre of the circle.	equal chords; equal angles
Equal chords in equal circles subtend equal angles at the circumference of the circles.	equal circles; equal chords; equal angles
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal angles
The opposite angles of a cyclic quadrilateral are supplementary	opposite angles of a cyclic quadrilateral
If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.	opposite angles of quad supplement OR converse opposite angles of a cyclic quadrilateral
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	exterior angle of a cyclic quadrilateral
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	exterior angle equal to the interior opposite angle OR converse exterior angle of a cyclic quadrilateral
Two tangents drawn to a circle from the same point outside the circle are equal in length	Tangents from common point OR Tangents from same point
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the endpoint of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem OR angle between line and chord
Quadrilaterals	
The interior angles of a quadrilateral add up to 360°.	sum of angles of a quadrilateral
The opposite sides of a parallelogram are parallel.	opposite sides of a parallelogram

If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opposite sides of a quadrilateral are parallel
The opposite sides of a parallelogram are equal in length.	opposite sides of a parallelogram
If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.	opposite sides of a quadrilateral are equal OR converse opposite sides of a parallelogram
The opposite angles of a parallelogram are equal.	opposite angles of a parallelogram
If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.	opposite angles of a quadrilateral are equal OR converse opposite angles of a parallelogram
The diagonals of a parallelogram bisect each other.	diagonals of a parallelogram
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diagonals of a quadrilateral bisect each other OR converse diagonals of a parallelogram
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opposite sides are equal and parallel
The diagonal of a parallelogram bisects its area.	Diagonal bisects area of a parallelogram
The diagonals of a rhombus bisect at right angles.	diagonals of a rhombus
The diagonals of a rhombus bisect the interior angles.	diagonals of a rhombus
All four sides of a rhombus are equal in length.	sides of a rhombus
All four sides of a square are equal in length.	sides of a square
The diagonals of a rectangle are equal in length.	diagonals of a rectangle
The diagonals of a kite intersect at right-angles.	diagonals of a kite
A diagonal of a kite bisects the other diagonal.	diagonal of a kite
A diagonal of a kite bisects the opposite angles	diagonal of a kite

It is important to note that:

A complete theorem statement can be used as a reason when answering problems on Euclidean geometry

PAPER A (Proof of theorems questions)

A1

QUESTION 7

7.1 Complete the statements below by filling in the missing word(s) so that the statements are CORRECT:

7.1.1 The angle subtended by a chord at the centre of a circle is … . (1)

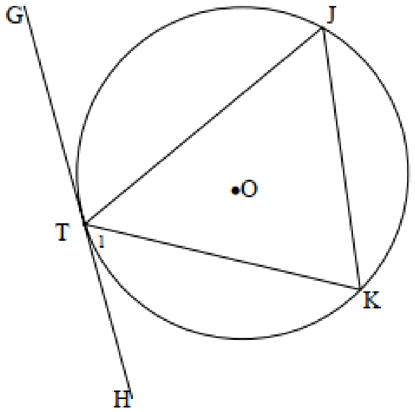
7.1.2 The angle between the tangent and a chord is … . (1)

7.1.3 The opposite angles of a cyclic quadrilateral are … . (1)

A2

QUESTION 8

8.1 In the diagram below O is the centre of the circle. GH is a tangent to the circle at T. J and K are points on the circumference of the circle. TJ, TK and JK are joined.



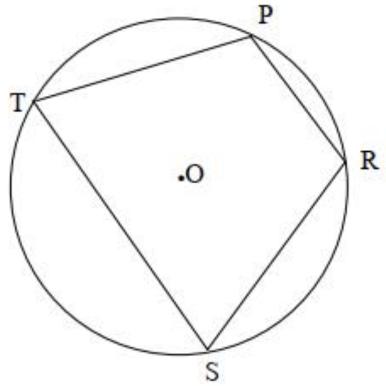
Prove the theorem that states $\hat{T}_1 = T\hat{J}K$. (5)

A3

QUESTION 9

9.1 In the figure below O is the centre of the circle and PRST is a cyclic quadrilateral.

Prove the theorem that states $P\hat{R}S + P\hat{T}S = 180°$.



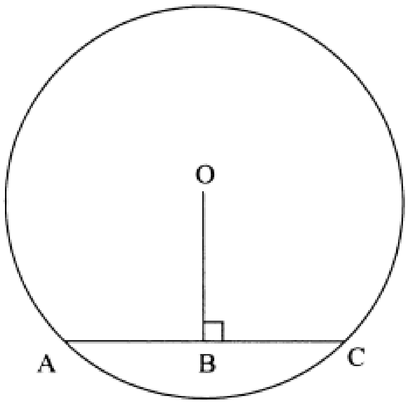
(5)

A4

QUESTION 11

In the diagram below, O is the centre of the circle and OB is perpendicular to the chord AC.

Prove, using Euclidean geometry methods, the theorem that states AB = BC.

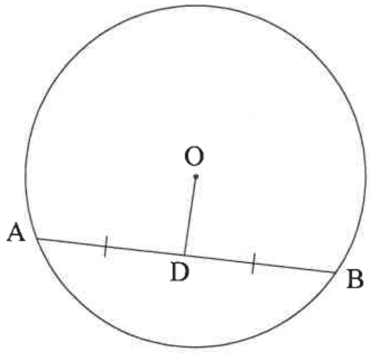


[5]

A5

QUESTION 9

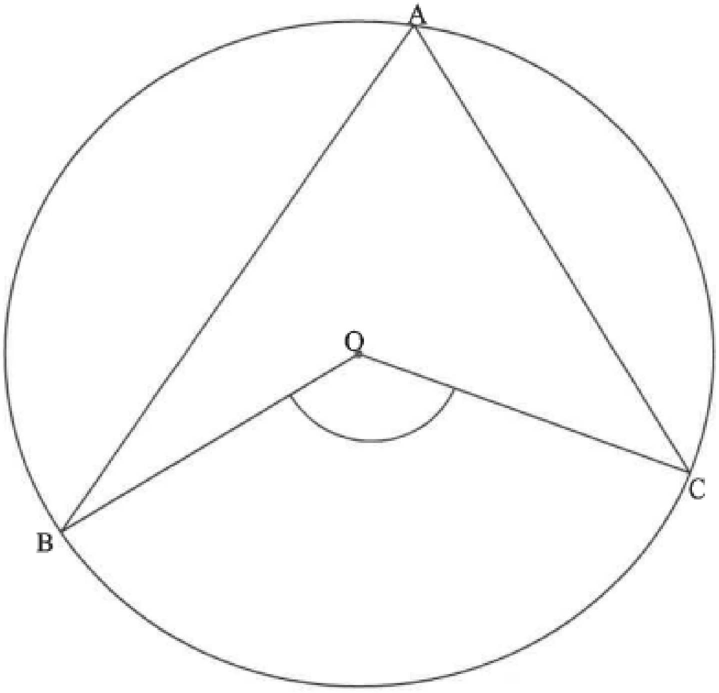
9.1 In the diagram, O is the centre of a circle. OD bisects chord AB.



A6

QUESTION 10

10.1 In the diagram, O is the centre of the circle with A, B and C drawn on the circle.

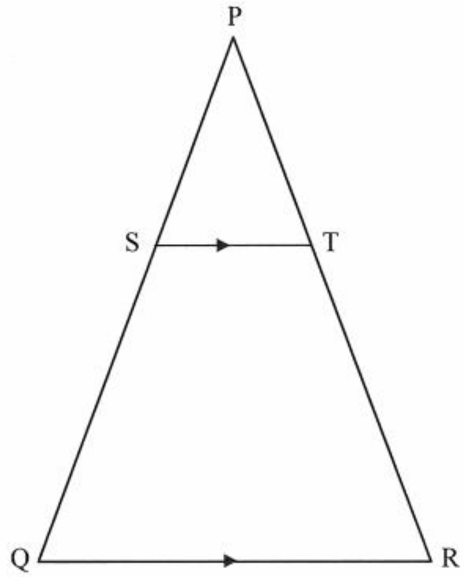


Prove the theorem which states that $B\hat{O}C = 2\hat{A}$. (5)

A7

QUESTION 10

10.1 In the diagram ΔPQR is drawn. S and T are points on sides PQ and PR respectively such that $ST \parallel QR$.

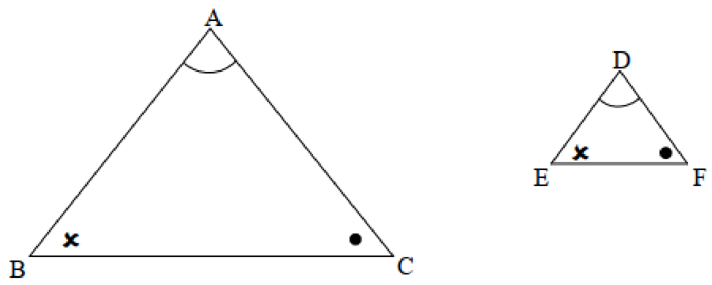


Prove the theorem which states that $\frac{PS}{SQ} = \frac{PT}{TR}$. (6)

A8

QUESTION 10

10.1 In the diagram, ΔABC and ΔDEF are drawn such that $\hat{A}=\hat{D}$, $\hat{B}=\hat{E}$ and $\hat{C}=\hat{F}$.

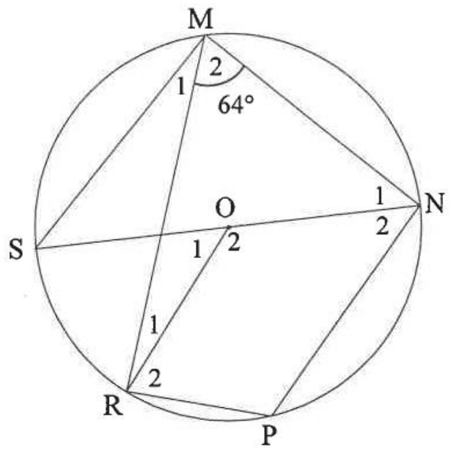


Use the diagram in the ANSWER BOOK to prove the theorem which states that if two triangles are equiangular, then the corresponding sides are in proportion, that is $\frac{AB}{DE}=\frac{AC}{DF}$. (6)

PAPER B

QUESTION 8

8.1 In the diagram, O is the centre of the circle. MNPR is a cyclic quadrilateral and SN is a diameter of the circle. Chord MS and radius OR are drawn. $\hat{M}_2 = 64°$.



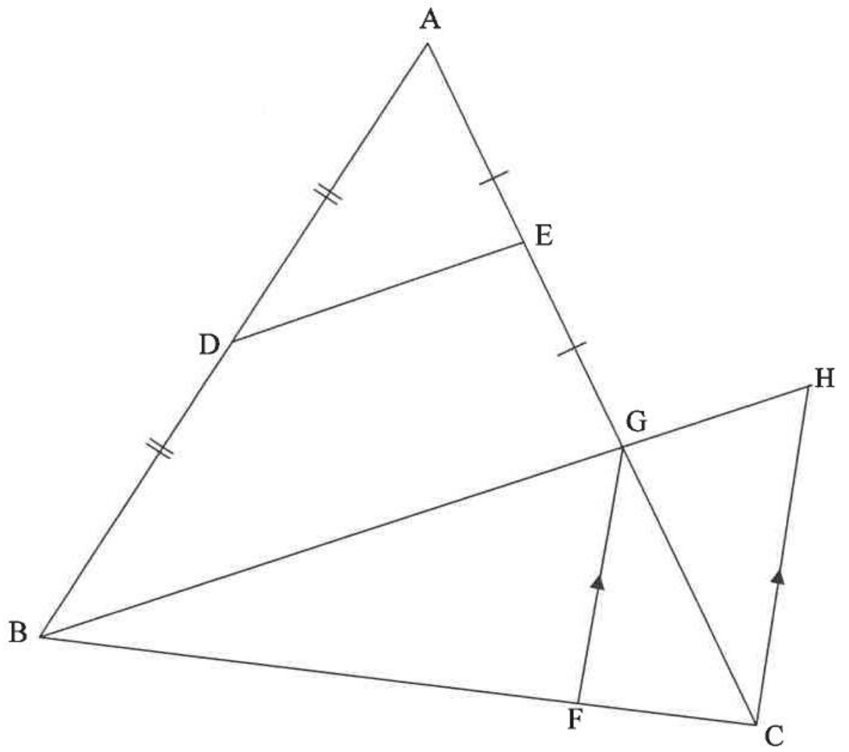
Determine, giving reasons, the size of the following angles:

8.1.1 $\hat{P}$ (2)

8.1.2 $\hat{M}_1$ (2)

8.1.3 $\hat{O}_1$ (2)

8.2 In the diagram, ΔABG is drawn. D and E are midpoints of AB and AG respectively. AG and BG are produced to C and H respectively. F is a point on BC such that FG || CH.

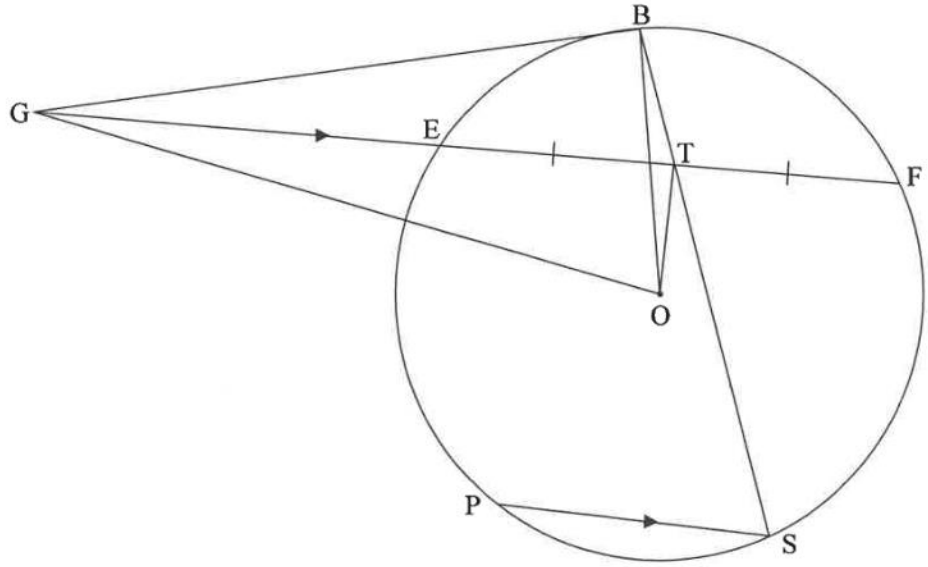


8.2.1 Give a reason why DE || BH. (1)

8.2.2 If it is further given that $\frac{FC}{BF} = \frac{1}{4}$, $DE = 3x - 1$ and $GH = x + 1$, calculate, giving reasons, the value of x. (6)

[13]

9.2 In the diagram, E, B, F, S and P are points on the circle centred at O. GB is a tangent to the circle at B. FE is produced to meet the tangent at G. OT is drawn such that T is the midpoint of EF. GO and BO are drawn. BS is drawn through T. PS || GF.



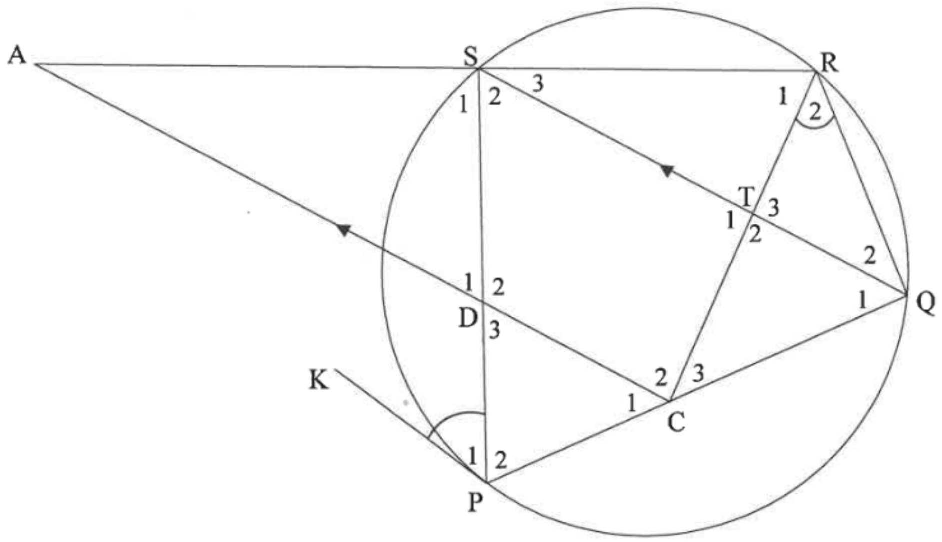
Prove, giving reasons, that:

9.2.1 OTBG is a cyclic quadrilateral (5)

9.2.2 $G\hat{O}B = \hat{S}$ (4)

QUESTION 10

In the diagram, PQRS is a cyclic quadrilateral. KP is a tangent to the circle at P. C and D are points on chords PQ and PS respectively and CD produced meets RS produced at A. CA || QS. RC is drawn. $\hat{P}_1 = \hat{R}_2$.



Prove, giving reasons, that:

10.1 $\hat{S}_1 = \hat{T}_2$ (4)

10.2 $\frac{AD}{AR} = \frac{AS}{AC}$ (5)

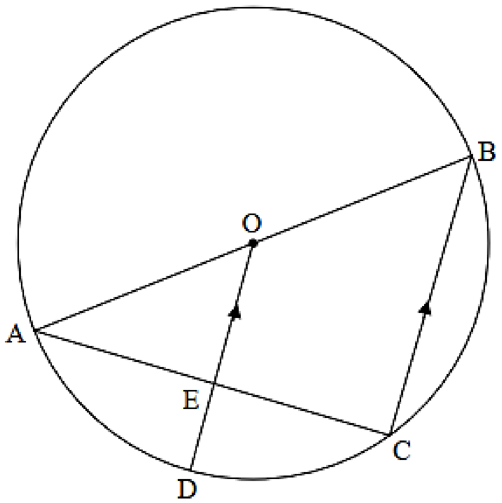
10.3 $AC \times SD = AR \times TC$ (4)

[13]

PAPER C

QUESTION 9

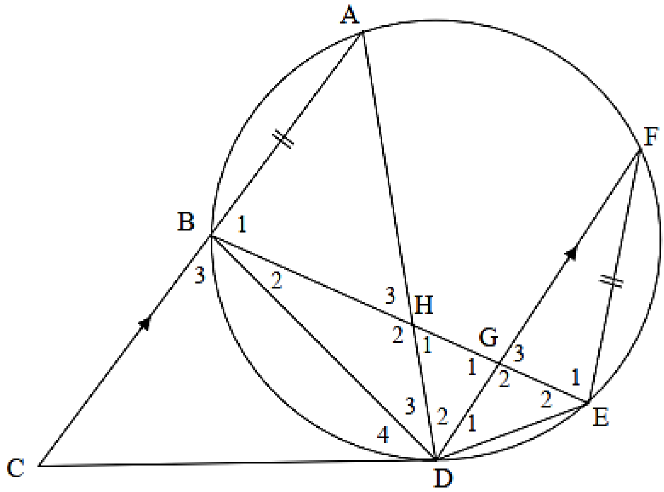
AB is a diameter of the circle ABCD. OD is drawn parallel to BC and meets AC in E.



If the radius is 10 cm and $AC = 16$ cm, calculate the length of ED. **[5]**

QUESTION 10

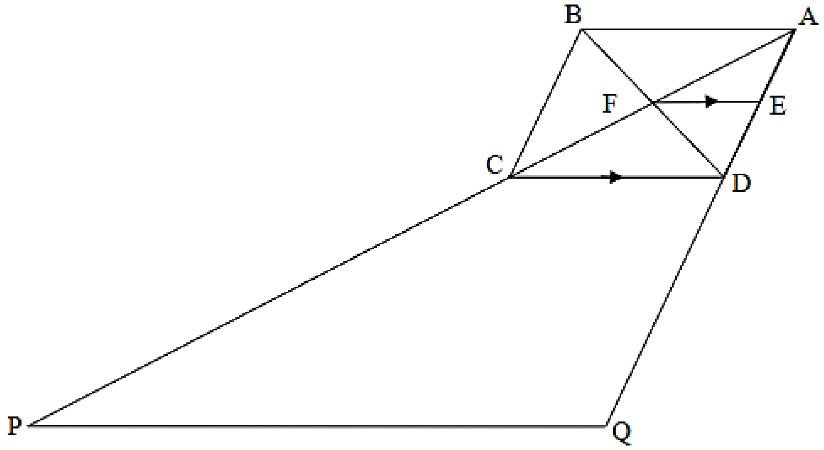
CD is a tangent to circle ABDEF at D. Chord AB is produced to C. Chord BE cuts chord AD in H and chord FD in G. AC || FD and FE = AB. Let $\hat{D}_4 = x$ and $\hat{D}_1 = y$.



10.1	Determine THREE other angles that are each equal to x.	(6)
10.2	Prove that $\Delta BHD \,|||\, \Delta FED$.	(5)
10.3	Hence, or otherwise, prove that $AB.BD = FD.BH$.	(2)

QUESTION 11

ABCD is a parallelogram with diagonals intersecting at F. FE is drawn parallel to CD. AC is produced to P such that $PC = 2AC$ and AD is produced to Q such that $DQ = 2AD$.



11.1 Show that E is the midpoint of AD. (2)

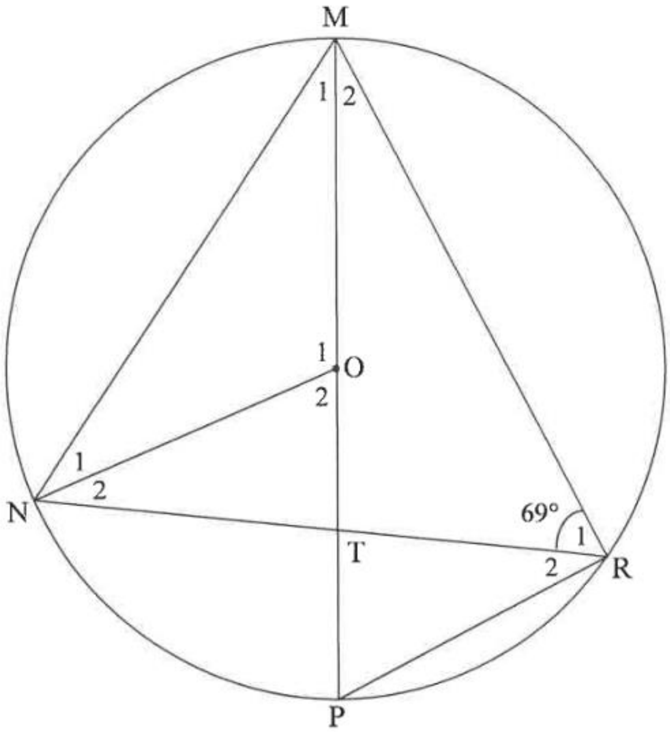
11.2 Prove $PQ \parallel FE$. (3)

11.3 If PQ is 60 cm, calculate the length of FE. (5)

PAPER D

QUESTION 8

8.1 In the diagram, MP is a diameter of a circle centered at O. MP cuts the chord NR at T. Radius NO and chords PR, MN and MR are drawn. $\hat{R}_1 = 69°$.



Determine, giving reasons, the size of:

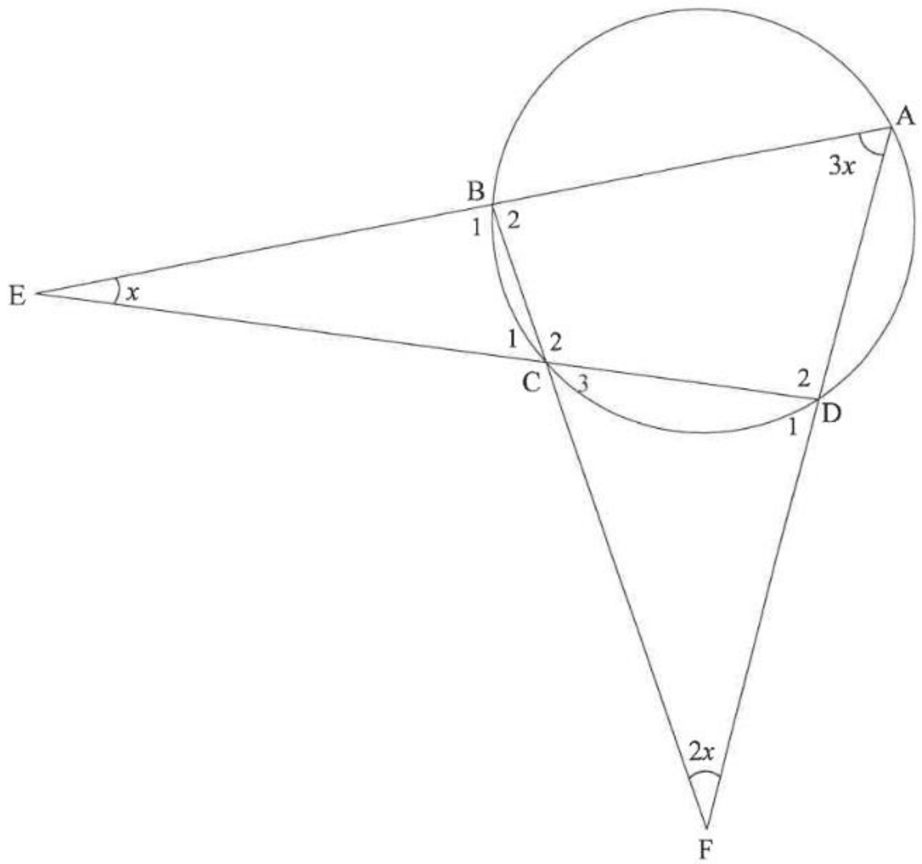
8.1.1 $\hat{R}_2$ (2)

8.1.2 $\hat{O}_1$ (2)

8.1.3 $\hat{M}_1$ (2)

8.1.4 $\hat{M}_2$, if it is further given that $NO \parallel PR$ (4)

8.2 In the diagram below, ABCD is a cyclic quadrilateral. AB and DC are produced to meet at E. AD and BC are produced to meet at F. $A\hat{F}B = 2x$, $D\hat{A}B = 3x$ and $A\hat{E}D = x$.

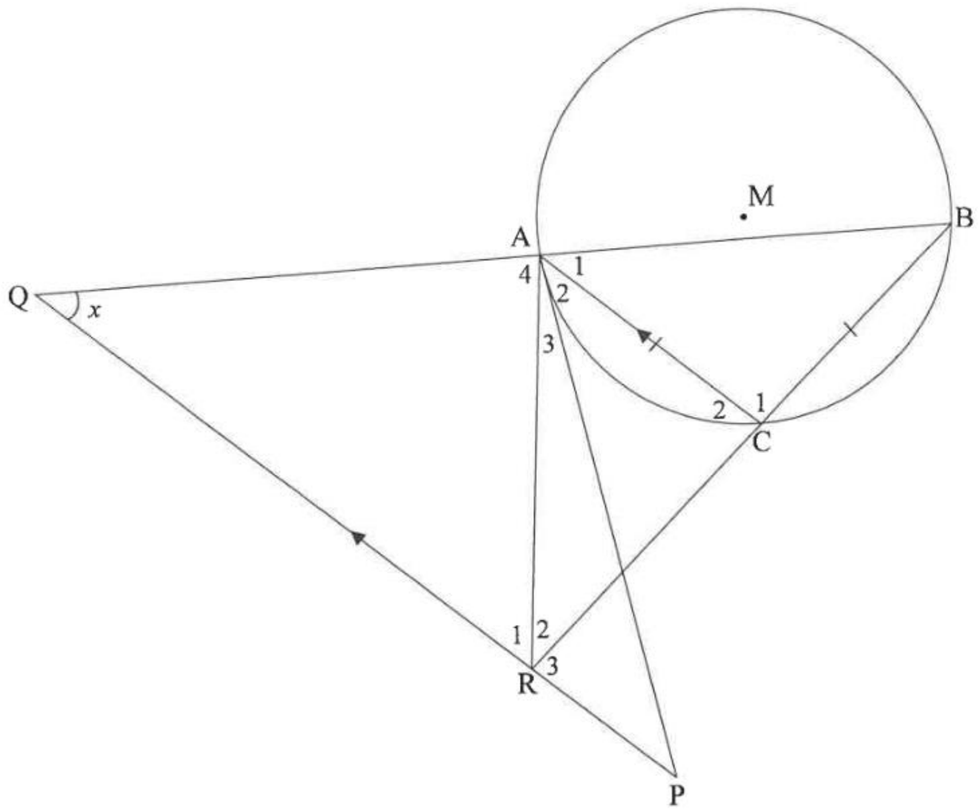


Determine, giving reasons, the value of x. (6)

[16]

QUESTION 9

9.2 In the diagram, M is the centre of the circle. A, B and C are points on the circle such that AC = BC. PA is a tangent to the circle at A. PQ is drawn parallel to CA to meet BA produced at Q. BC produced meets PQ at R and AR is drawn. Let $\hat{Q} = x$.



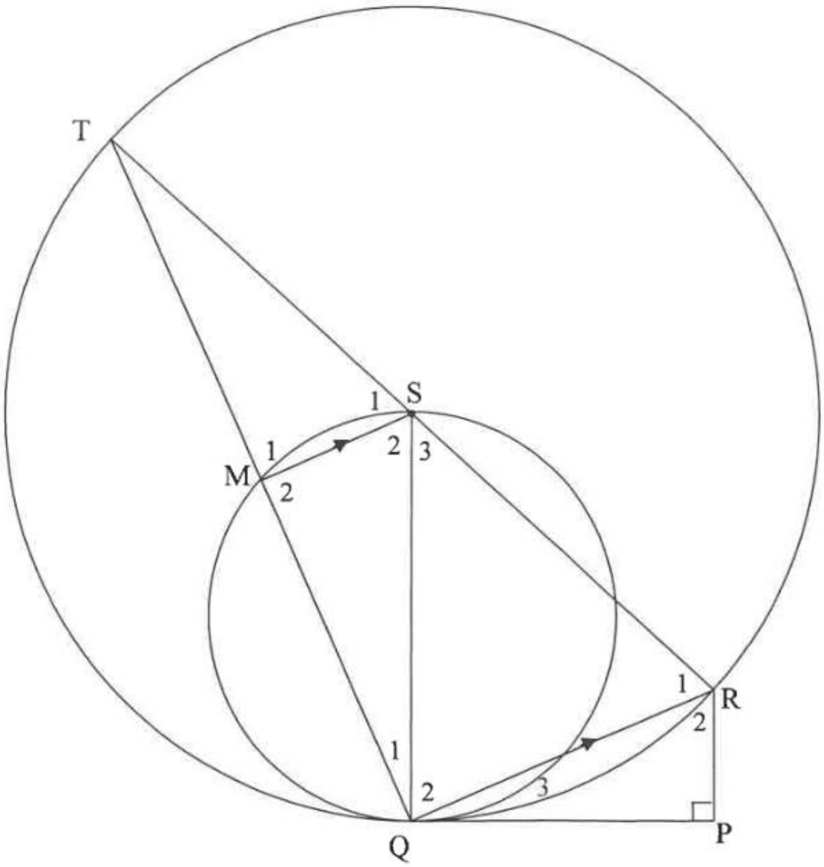
9.2.1 Determine, giving reasons, FOUR other angles EACH equal to x. (6)

9.2.2 Prove that ABPR is a cyclic quadrilateral. (2)

9.2.3 Prove that $\frac{BA}{BQ} = \frac{BC}{QR}$. (3)

QUESTION 10

In the diagram, TSR is a diameter of the larger circle having centre S. Chord TQ of the larger circle cuts the smaller circle at M. PQ is a common tangent to the two circles at Q. SQ is drawn.
$RP \perp PQ$ and $MS \parallel QR$.



10.1 Prove, giving reasons that:

10.1.1 SQ is the diameter of the smaller circle (3)

10.1.2 $RT = \dfrac{RQ^2}{RP}$ (6)

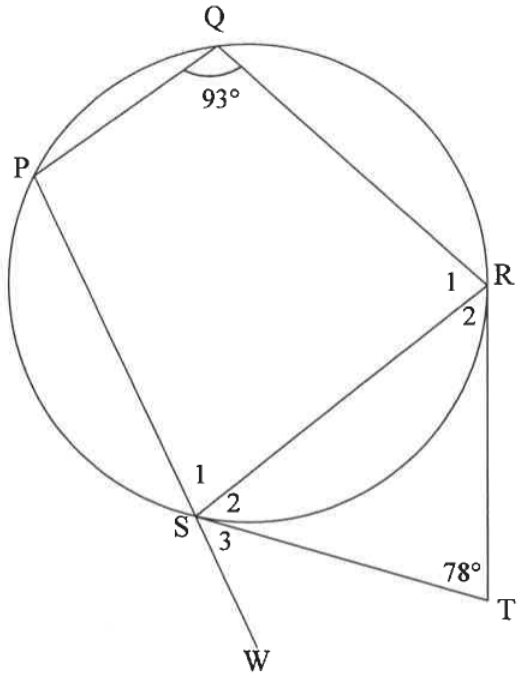
10.2 If $MS = 14$ units and $PQ = \sqrt{640}$ units, calculate, giving reasons, the length of the radius of the larger circle. (6)

[15]

PAPER E

QUESTION 9

In the diagram, PQRS is a cyclic quadrilateral. PS is produced to W. TR and TS are tangents to the circle at R and S respectively. $\hat{T} = 78°$ and $\hat{Q} = 93°$.



9.1 Give a reason why ST = TR. (1)

9.2 Calculate, giving reasons, the size of:

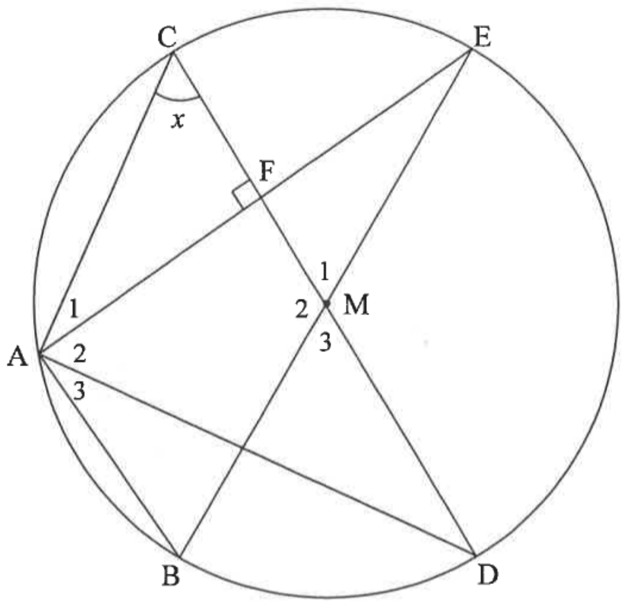
9.2.1 $\hat{S}_2$ (2)

9.2.2 $\hat{S}_3$ (2)

[5]

QUESTION 10

In the diagram, BE and CD are diameters of a circle having M as centre. Chord AE is drawn to cut CD at F. $AE \perp CD$. Let $\hat{C} = x$.



10.1 Give a reason why AF = FE. (1)

10.2 Determine, giving reasons, the size of $\hat{M}_1$ in terms of x. (3)

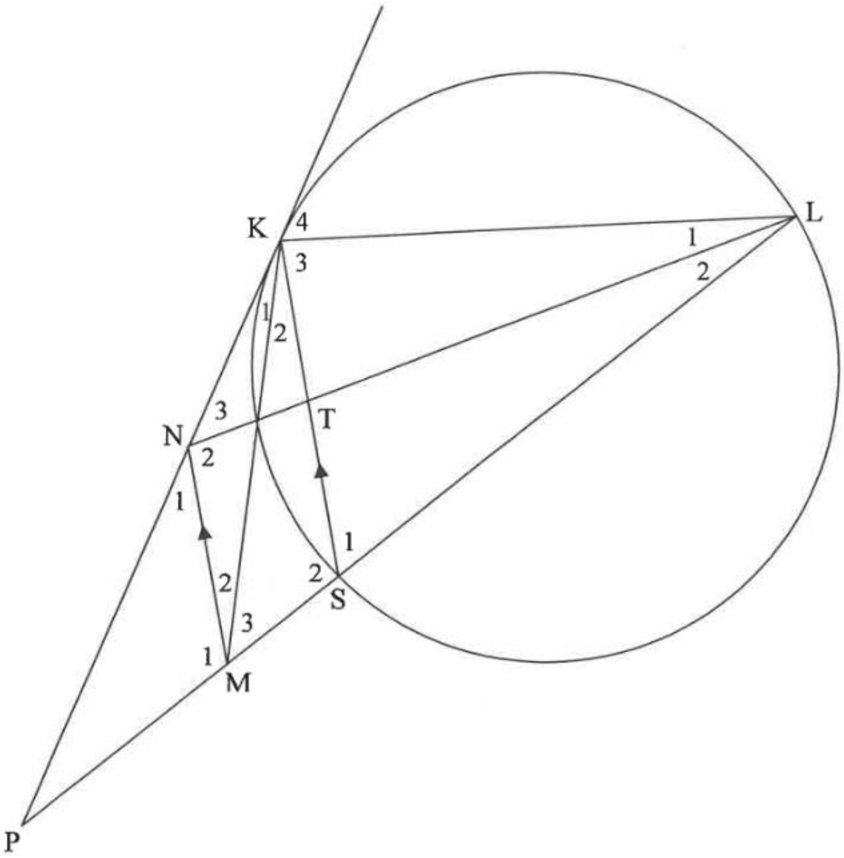
10.3 Prove, giving reasons, that AD is a tangent to the circle passing through A, C and F. (4)

10.4 Given that CF = 6 units and AB = 24 units, calculate, giving reasons, the length of AE. (5)

[13]

QUESTION 11

11.2 In the diagram, PK is a tangent to the circle at K. Chord LS is produced to P. N and M are points on KP and SP respectively such that MN || SK. Chord KS and LN intersect at T.



11.2.1 Prove, giving reasons, that:

(a) $\hat{K}_4 = N\hat{M}L$ (4)

(b) KLMN is a cyclic quadrilateral (1)

11.2.2 Prove, giving reasons, that $\Delta LKN \,|||\, \Delta KSM$. (5)

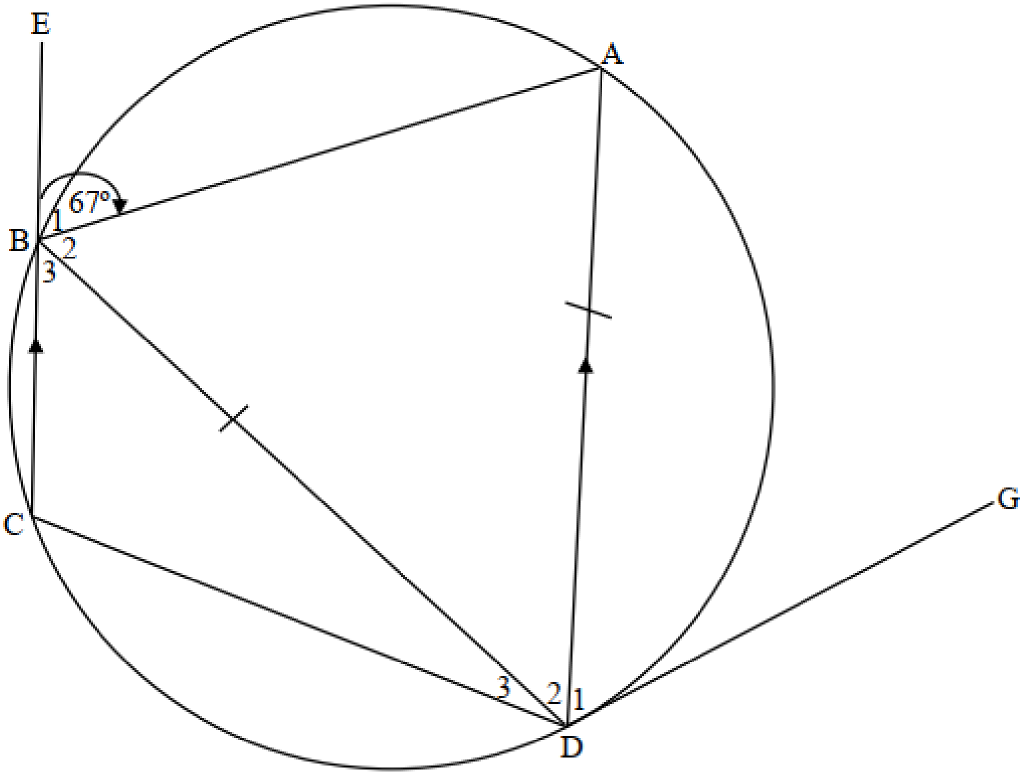
11.2.3 If $LK = 12$ units and $3KN = 4SM$, determine the length of KS. (4)

11.2.4 If it is further given that $NL = 16$ units, $LS = 13$ units and $KN = 8$ units, determine, with reasons, the length of LT. (4)

PAPER F

QUESTION 8

In the diagram below, points A, B, C and D lie on the circumference of a circle with AD || EC. CB is produced to E. GD is a tangent to the circle at D and DB = AD.
$E\hat{B}A = 67°$.



8.1 Calculate, with reasons, the size of the following angles:

8.1.1 $A\hat{D}C$ (2)

8.1.2 $\hat{C}$ (1)

8.1.3 $\hat{A}$ (1)

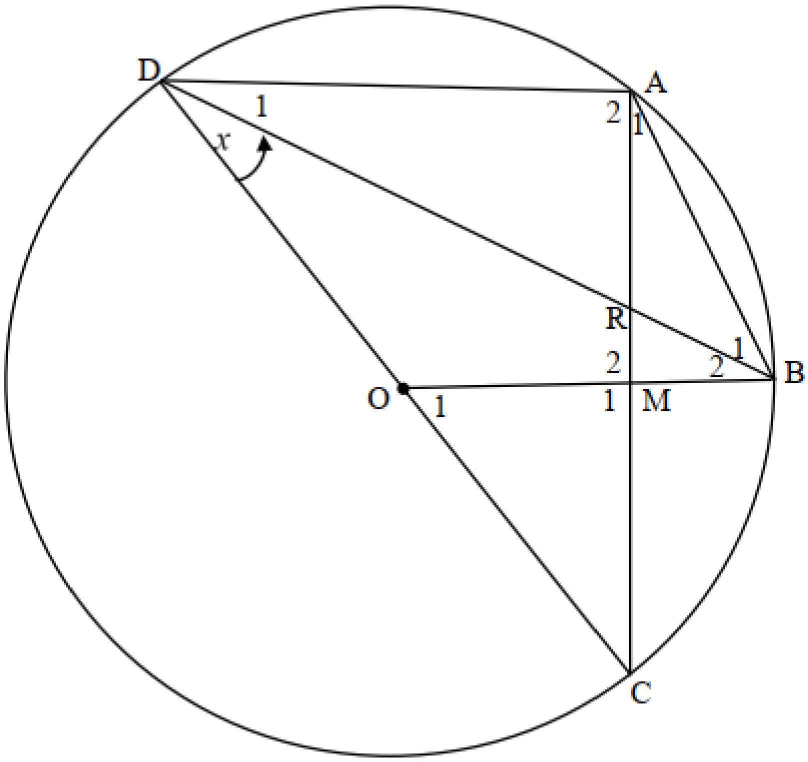
8.1.4 $\hat{D}_2$ (3)

8.1.5 $B\hat{D}G$ (2)

8.2 Prove that AB = CD. (2)

QUESTION 9

9.1 In the diagram below, A, B, C and D are points on a circle with centre O. OB intersects AC at M, the midpoint of chord AC.
Let $B\hat{D}C = x$.



9.1.1 Determine, with reasons, in terms of x:

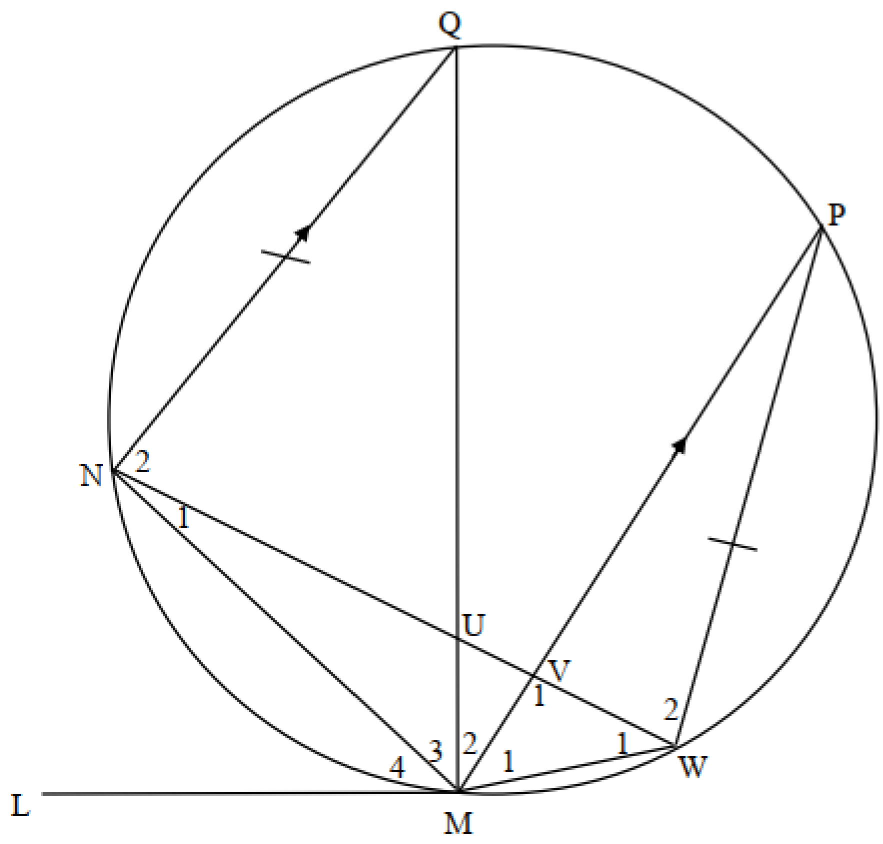
(a) $\hat{O}_1$ (1)

(b) $A\hat{B}O$ (4)

9.1.2 Prove that AB is a tangent to the circle that passes through points A, D and R. (6)

9.1.3 Prove that $AD^2 = 4DO^2 - 4AB^2 + 4MB^2$. (4)

9.2 In the diagram below, LM is a tangent to circle QNMWP at M. NW cuts QM and PM at U and V respectively.
NQ = WP and NQ || MP.

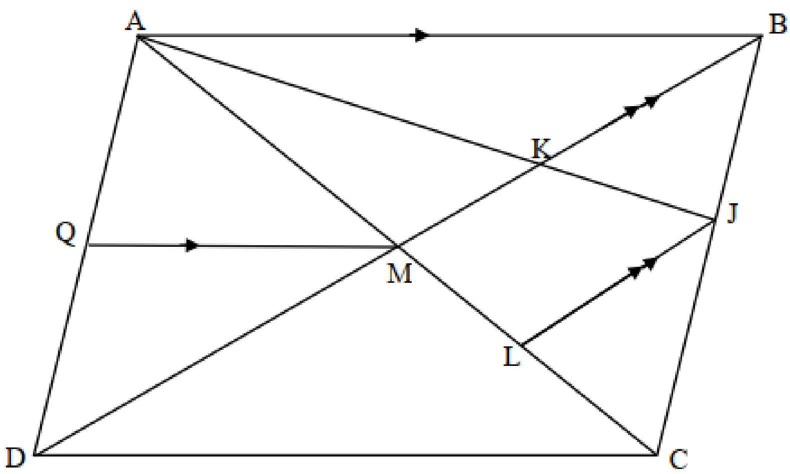


9.2.1 State, with reasons, THREE angles equal to $\hat{M}_2$. (3)

9.2.2 Prove that $\Delta WMV ||| \Delta QMN$. (3)

9.2.3 Prove that $\dfrac{MV}{WV} = \dfrac{MN}{PW}$. (3)

10.2 ABCD is a parallelogram with diagonals that intersect at M. J is a point on BC. BJ : JC is 2 : 3. AJ meets BD at K. BD || JL and JL meets AC at L. Q is a point on AD such that AB || QM.



10.2.1 Determine, with reasons, the following ratios:

(a) $\frac{ML}{LC}$ (2)

(b) $\frac{AK}{KJ}$ (3)

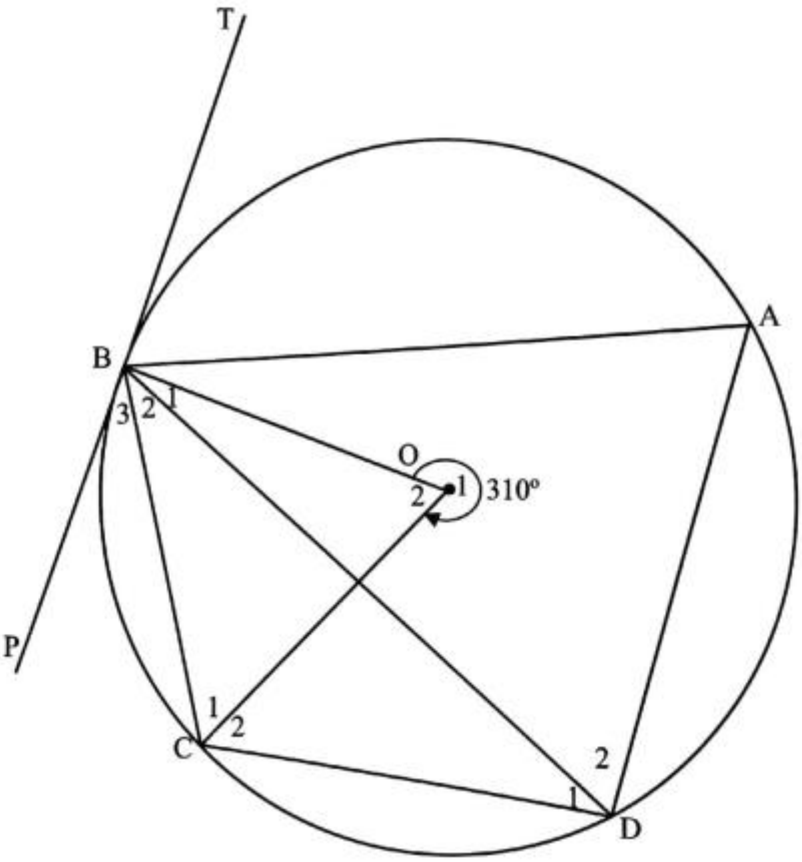
10.2.2 If $AB = \sqrt{10}$ units and $BC = \frac{2}{3}AB$.

Calculate the length of AQ. (4)

PAPER G

QUESTION 8

In the diagram below, A, B, C and D are points on a circle having centre O.
PBT is a tangent to the circle at B.
Reflex $B\hat{O}C = \hat{O}_1 = 310°$ as shown in the diagram below.



Calculate, giving reasons, the size of:

8.1 $\hat{D}_1$ (3)

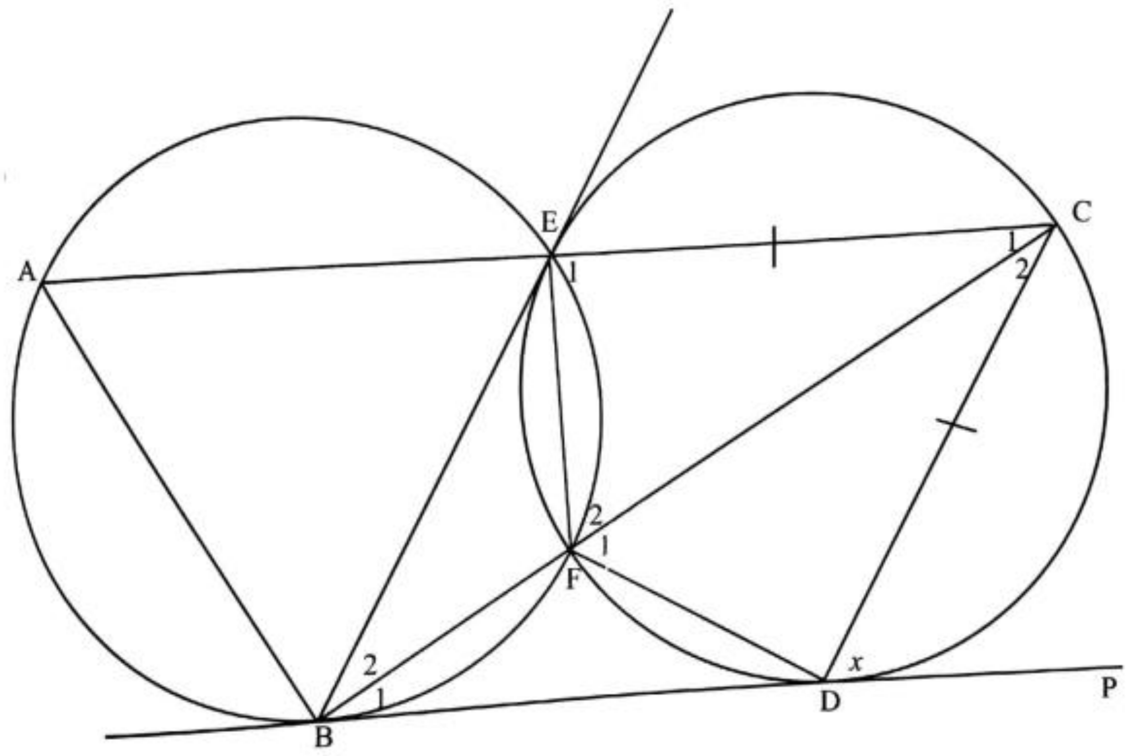
8.2 $\hat{B}_3$ (2)

8.3 $\hat{B}_1$, if it is given that $\hat{A} = 60°$. (4)

QUESTION 9

9.1 Complete the statement so that it is TRUE.
Angles subtended by a chord of a circle, on the same side of a chord, are … (1)

9.2 In the diagram below, ABFE and EFDC are cyclic quadrilaterals in two equal circles that intersect at E and F. BFC and AEC are straight lines. BD is a common tangent to the circles at B and D respectively. EC = CD.
Let $C\hat{D}P = x$



Prove, giving reasons, that:

9.2.1 $\hat{F}_1 = \hat{F}_2$. (3)

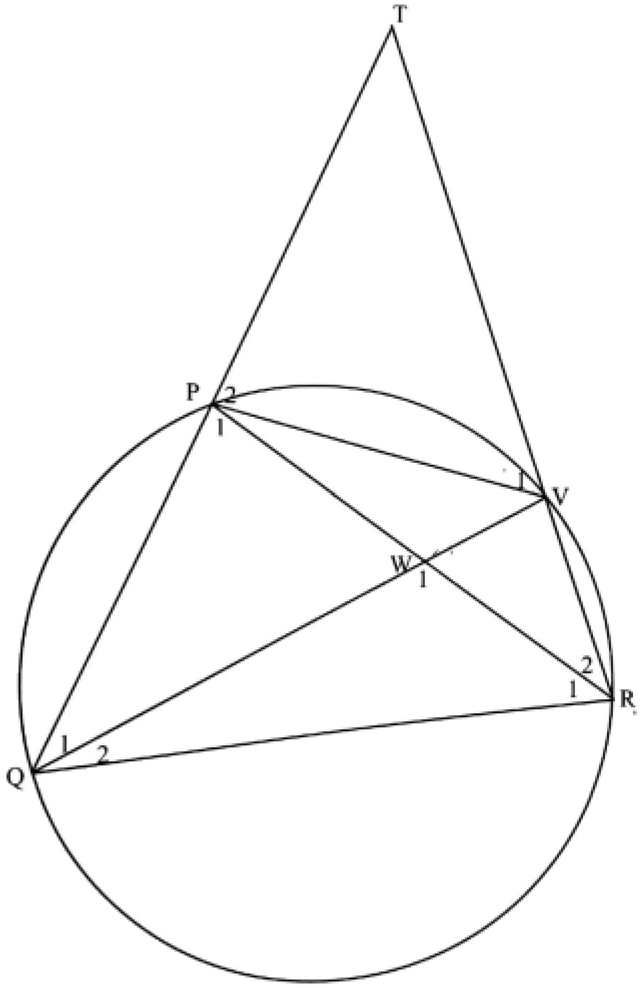
9.2.2 ABDC is a cyclic quadrilateral. (3)

9.2.3 BE || CD. (2)

9.2.4 FC is a diameter of circle FDCE if it is given that EBDC is a rhombus. (5)

Question 10

10.2 In the diagram below, ΔPQR is an equilateral triangle inscribed in a circle. V is a point on the circle. QP produced meets RV produced at T. PR and QV intersect at W.



Prove, giving reasons, that:

10.2.1 $\hat{W}_1 = T\hat{R}Q$ (3)

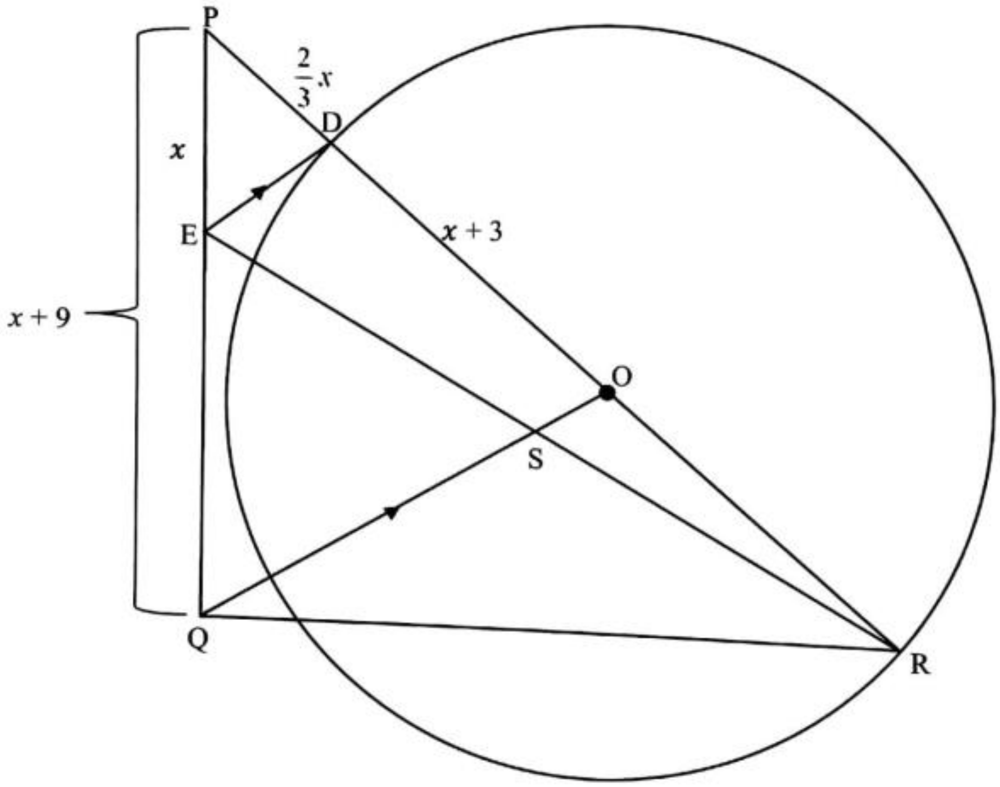
10.2.2 $\Delta TQR \,|||\, \Delta QRW$ (3)

10.2.3 $\dfrac{PT}{QW} = \dfrac{PV}{WR}$ (6)

QUESTION 11

In the diagram below, the circle with centre O is drawn. OQ is drawn parallel to a tangent to the circle at D. ER is drawn with S on OQ.
RD is produced to P and PQ is joined.

PE = x units, PQ = $x + 9$ units, PD = $\frac{2}{3}x$ units and DO = $x + 3$ units.

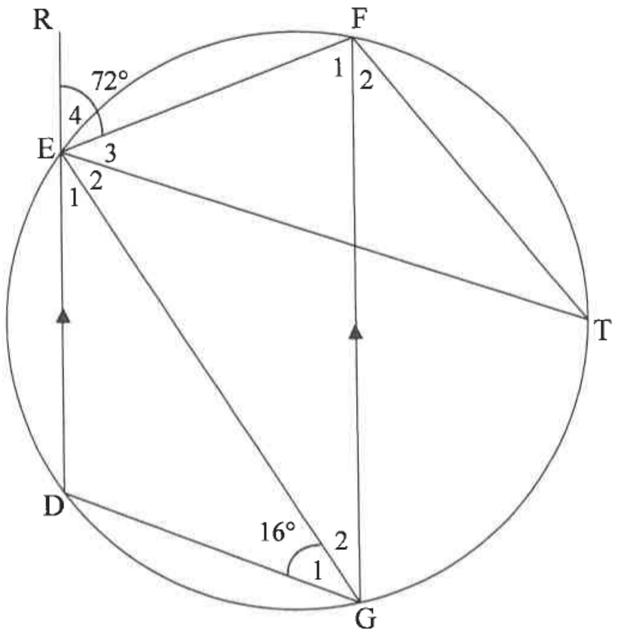


11.1	Calculate the length of RO.	(4)
11.2	If OS = 1,4 units and S is the midpoint of ER, determine the length of DE.	(2)
11.3	If the area of $\Delta PED = 2{,}7$ units^2, find the area of ΔPER.	(4)

PAPER H

QUESTION 9

9.1 In the diagram, DEFG is a cyclic quadrilateral with DE || GF. DE is produced to R. T is another point on the circle. EG, FT and ET are drawn. $\hat{E}_4 = 72°$ and $\hat{G}_1 = 16°$.



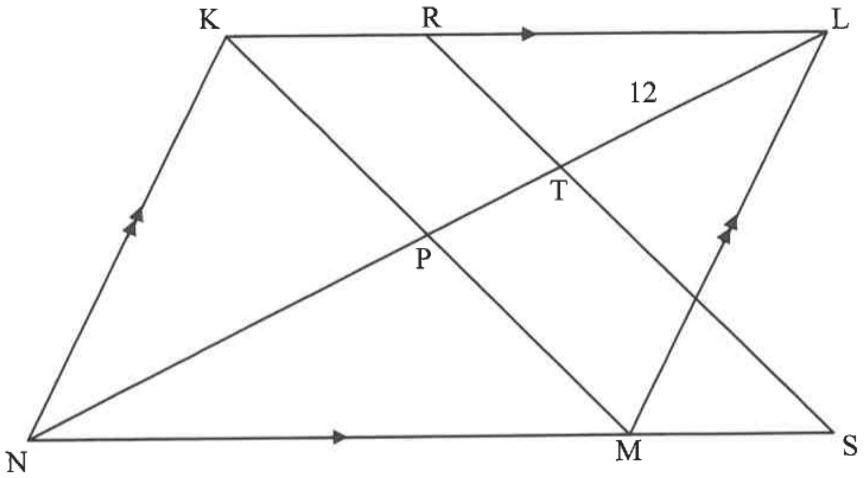
Determine, with reasons, the size of the following angles:

9.1.1 $D\hat{G}F$ (2)

9.1.2 $\hat{T}$ (2)

9.1.3 $G\hat{E}F$ (2)

9.2 In the diagram, the diagonals of parallelogram KLMN intersect at P. NM is produced to S. R is a point on KL and RS cuts PL at T.
NM : MS = 4 : 1, NL = 32 units and TL = 12 units.



9.2.1 Determine, with reasons, the value of the ratio NP : PT in simplest form. (4)

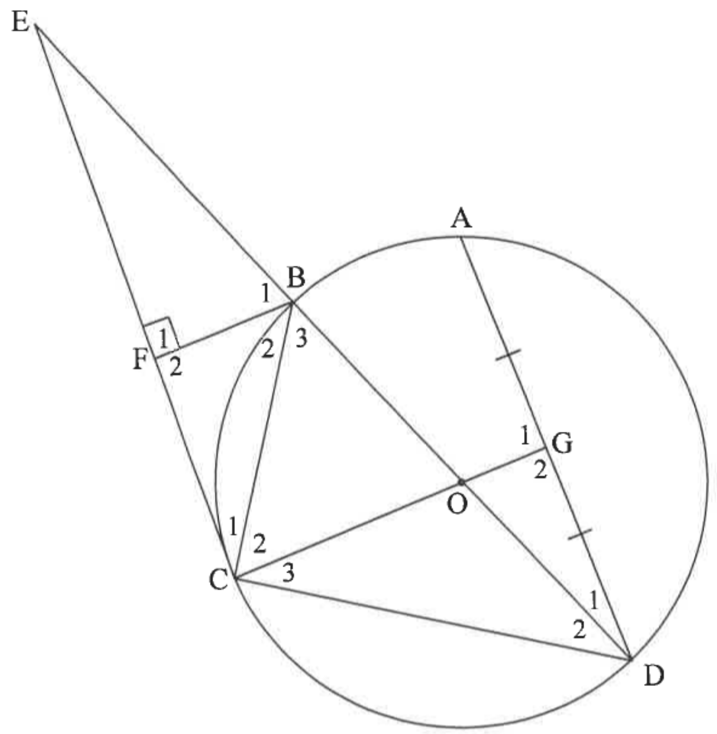
9.2.2 Prove, with reasons, that KM || RS. (2)

9.2.3 If NM = 21 units, determine, with reasons, the length of RL. (4)

[16]

QUESTION 10.2

10.2 In the diagram, O is the centre of a circle passing through A, B, C and D. EC is a tangent to the circle at C. Diameter DB produced meets tangent EC at E. F is a point on EC such that BF ⊥ EC. Radius CO produced bisects AD at G. BC and CD are drawn.



10.2.1 Prove, with reasons, that:

(a) FB || CG (3)

(b) ΔFCB ||| ΔCDB (5)

10.2.2 Give a reason why $\hat{G}_1 = 90°$. (1)

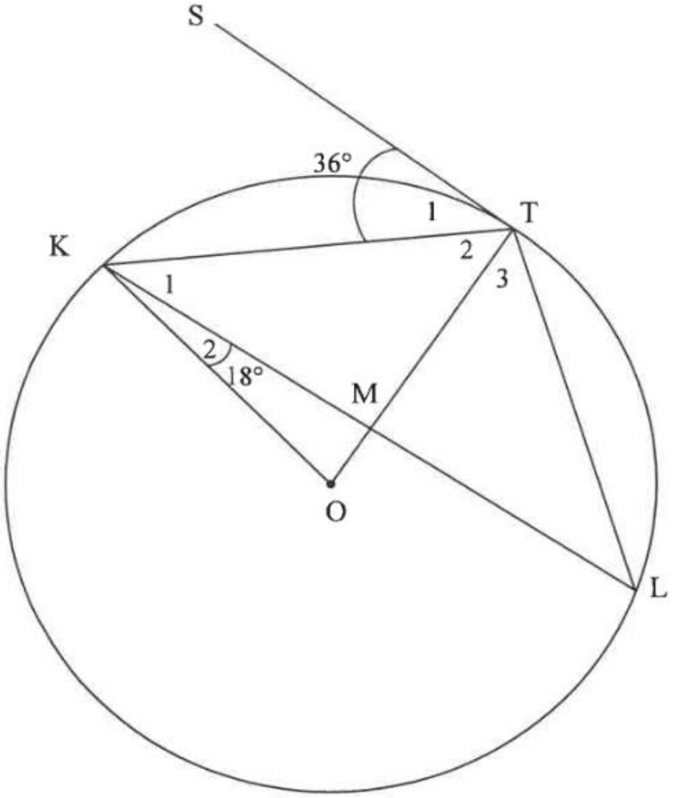
10.2.3 Prove, with reasons, that $CD^2 = CG.DB$. (5)

10.2.4 Hence, prove that $DB = CG + FB$. (5)

PAPER I

QUESTION 8

8.1 In the diagram, O is the centre of the circle. K, T and L are points on the circle. KT, TL, KL, OK and OT are drawn. OT intersects KL at M. ST is a tangent to the circle at T. $S\hat{T}K = 36°$ and $O\hat{K}L = 18°$.



8.1.1 Determine, giving reasons, the size of:

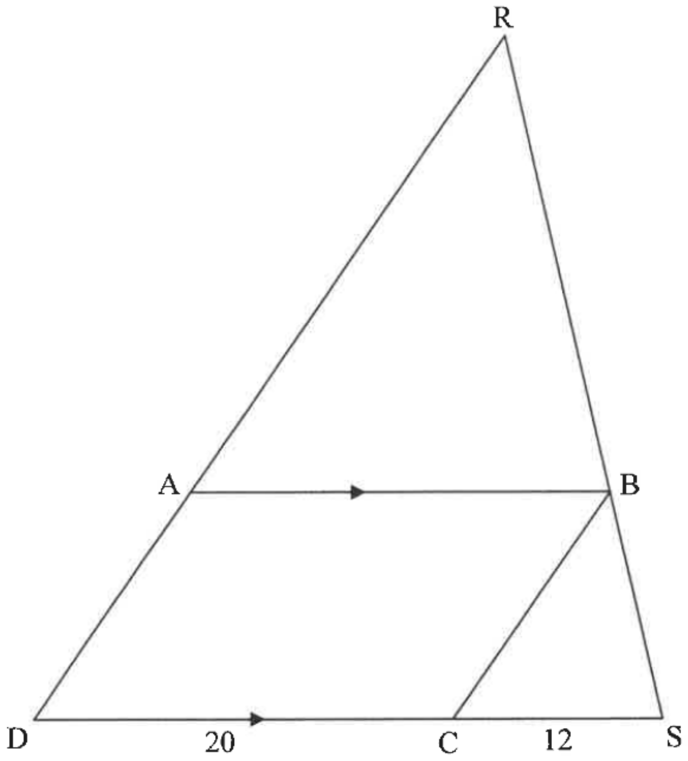
(a) $\hat{T}_2$ (2)

(b) $\hat{L}$ (2)

(c) $K\hat{O}T$ (2)

8.1.2 Prove, giving reasons, that $KM = ML$. (3)

8.2 In the diagram, ΔRDS is drawn. A, B and C are points on RD, RS and DS respectively such that $AB \parallel DS$ and $RB : BS = 5 : 3$. $DC = 20$ units and $CS = 12$ units.



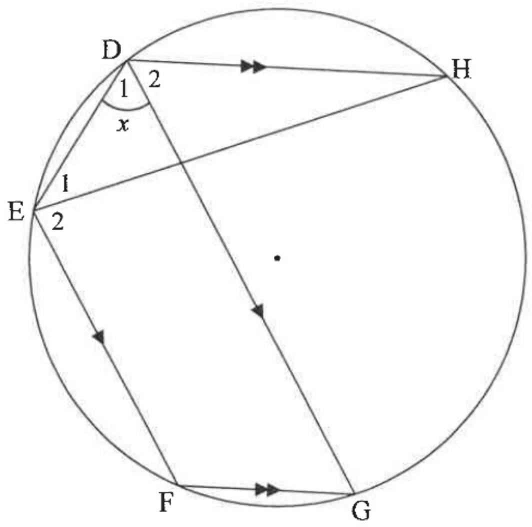
8.2.1 Prove, giving reasons, that $BC \parallel AD$. (3)

8.2.2 If it is further given that $RD = 48$ units, calculate, giving reasons, the value of the ratio $AD : AB$. (3)

[15]

QUESTION 9.2

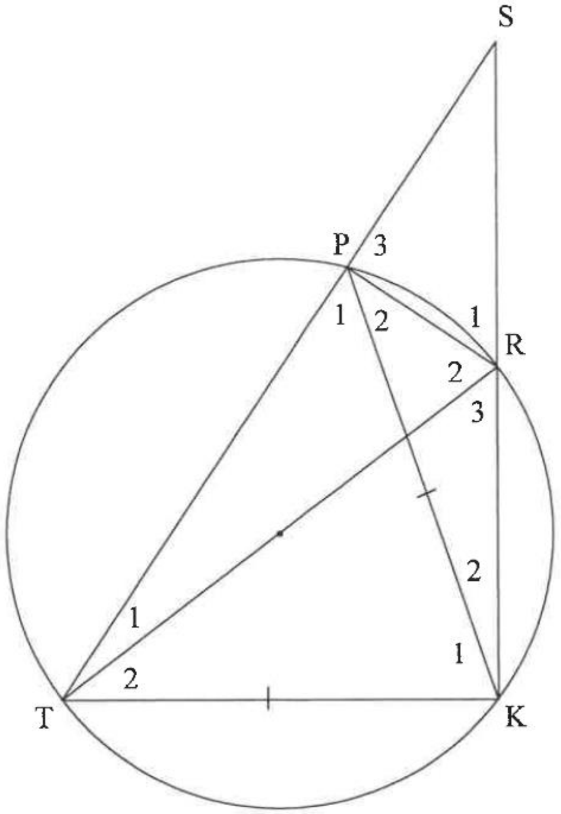
9.2 In the diagram, DEFG is a cyclic quadrilateral such that EF || DG. H is another point on the circle such that DH || FG. Chord EH is drawn. Let $\hat{D}_1 = x$.



Prove, giving reasons, that $\hat{D}_1 = \hat{D}_2$. (4)

QUESTION 10

In the diagram, TR is a diameter of the circle. PRKT is a cyclic quadrilateral. Chords TP and KR are produced to intersect at S. Chord PK is drawn such that PK = TK.



10.1 Prove, giving reasons, that:

10.1.1 SR is a diameter of a circle passing through points S, P and R (4)

10.1.2 $\hat{S} = \hat{P}_2$ (5)

10.1.3 $\Delta SPK \,|||\, \Delta PRK$ (3)

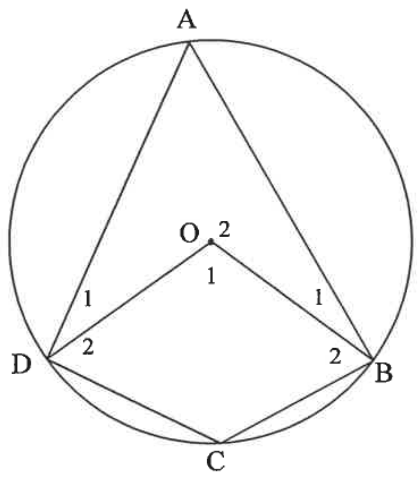
10.2 If it is further given that SR = RK, prove that $ST = \sqrt{6}RK$. (5)

[17]

PAPER J

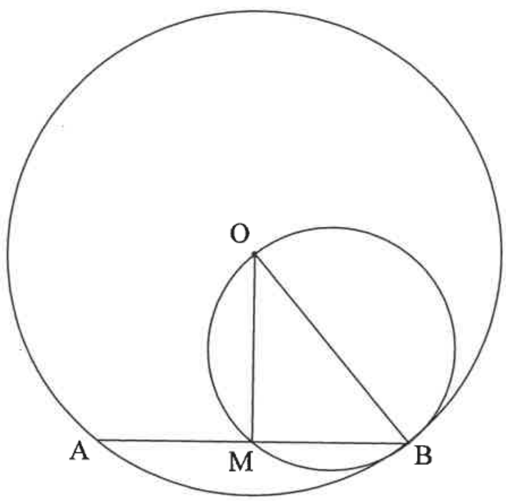
QUESTION 8.2 & 8.3

8.2 In the diagram, O is the centre of the circle and ABCD is a cyclic quadrilateral. OB and OD are drawn.



If $\hat{O}_1 = 4x + 100°$ and $\hat{C} = x + 34°$, calculate, giving reasons, the size of x. (5)

8.3 In the diagram, O is the centre of the larger circle. OB is a diameter of the smaller circle. Chord AB of the larger circle intersects the smaller circle at M and B.

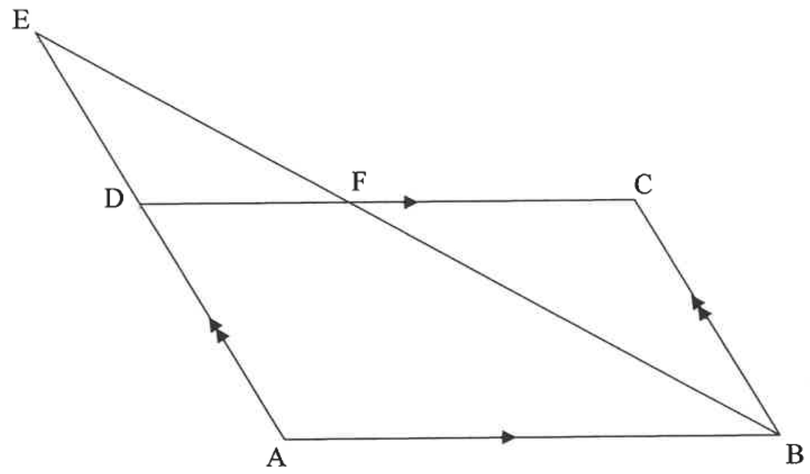


8.3.1 Write down the size of $O\hat{M}B$. Provide a reason. (2)

8.3.2 If $AB = \sqrt{300}$ units and $OM = 5$ units, calculate, giving reasons, the length of OB. (4)

QUESTION 9

In the diagram, ABCD is a parallelogram with AB = 14 units. AD is produced to E such that AD : DE = 4 : 3. EB intersects DC in F. EB = 21 units.



9.1 Calculate, with reasons, the length of FB. (3)

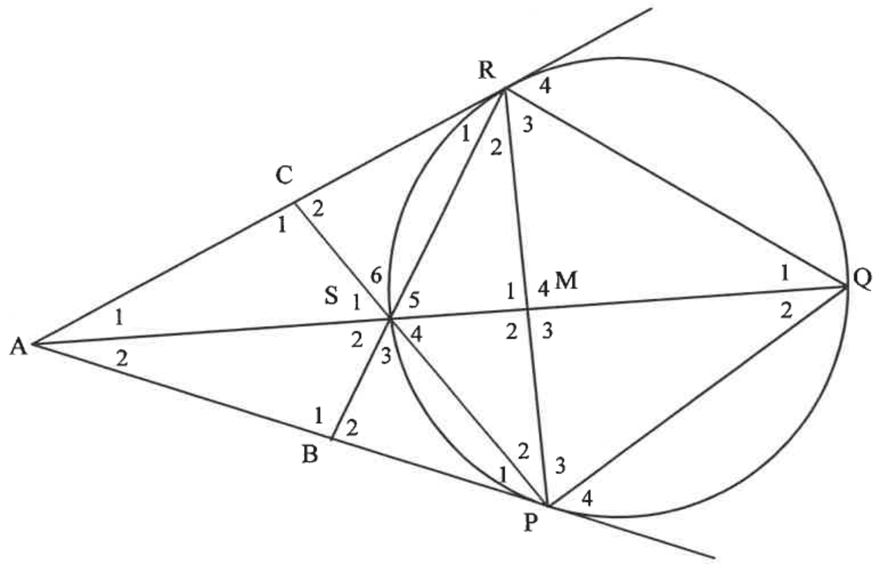
9.2 Prove, with reasons, that ΔEDF ||| ΔEAB. (3)

9.3 Calculate, with reasons, the length of FC. (3)

[9]

QUESTION 10

In the diagram, PQRS is a cyclic quadrilateral such that PQ = PR. The tangents to the circle through P and R meet QS produced at A. RS is produced to meet tangent AP at B. PS is produced to meet tangent AR at C. PR and QS intersect at M.



Prove, giving reasons, that:

10.1 $\hat{S}_3 = \hat{S}_4$ (5)

10.2 SMRC is a cyclic quadrilateral (4)

10.3 RP is a tangent to the circle passing through P, S and A at P (6)

[15]

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